Proof-Theoretic Semantics and Inquisitive Logic

Will Stafford

December 19, 2020

Abstract

Prawitz (1971) conjectured that proof-theoretic validity offers a semantics for intuitionistic logic. This conjecture has recently been proven false by Piecha and Schroeder-Heister (2019). This article resolves one of the questions left open by this recent result by showing the extensional alignment of proof-theoretic validity and general inquisitive logic. General inquisitive logic is a generalisation of the logic of inquisitive semantics, a uniform semantics for questions and assertions. The paper further defines a notion of quasi-proof-theoretic validity by restricting proof-theoretic validity to allow double negation elimination for atomic formulas and proves the extensional alignment of quasi-proof-theoretic validity and inquisitive logic.

Contents

1 Introduction

Proof-Theoretic Validity was proposed by Prawitz (1971) as an explication of Gentzen's famous claim that the introduction rules can be viewed as definitions of the connectives.[1] Prawitz's formal definition of proof-theoretic validity (henceforth PTV) is complex but, roughly, a proof is valid if it stands in the correct relationship to the introduction rules (see Definition 4.8). We can call the set of all formulas with proof-theoretically valid proofs the logic of PTV. Prawitz conjectured in the early 1970s that:

Conjecture 1.1 (Prawitz)**.** The logic of PTV is intuitionistic logic.

This conjecture remained open for many years but was recently disproven:

Theorem 1.2 (Piecha and Schroeder-Heister 2019, Corollary 3.9)**.** The logic of PTV is a proper superset of intuitionistic logic.

For those still sympathetic to Gentzen and Prawitz, this leaves open the question of what super-intuitionistic logic is the logic of PTV, and hence how far from true Prawitz's conjecture is.

Let quasi-PTV be PTV with double-negation elimination for atomic formulas.[2] The main result of this paper is:

Theorem 1.3. The logic of PTV is general inquisitive logic and the logic of quasi-PTV is inquisitive logic.

Inquisitive logic has been studied extensively in recent years, see for example Ciardelli and Roelofsen (2011), Ciardelli, Groenendijk, and Roelofsen (2018), and Punčochář (2016). It arises naturally out of an effort to capture the idea that propositions have both informative and inquisitive content.

The main result shows that inquisitive logic and general inquisitive logic have proof-theoretic semantics as well as model-theoretic and algebraic. It also shows which logic would be justified by proof-theoretic validity. Further, quasi-PTV is as classical as one can make PTV since inquisitive logic is the maximal weak intermediate logic with the disjunction property and double negation elimination holding for atomic formulas (Ciardelli and Roelofsen 2011, p. 18). So, adding more instances of double negation elimination will either not change the logic or the resulting system will be stronger than classical logic.

Given this, it is natural to wonder about the connection between PTV and inquisitive logic discovered here, and whether anything in the underlying motivations for these logics may be responsible for this. There are two things worth mentioning in this connection. First, one traditional semantics for intuitionistic logic is Kolmogorov's problem interpretation. This interpretation takes formulas as stating problems in need of a solution. For example, Kolmogorov (1932, p. 329) states that $a \vee b$ represents the problem "solve at least one of the problems a and b". While Kolmogorov had a broad range of problems in mind, including those that involve constructions, we can identify many problems with the questions they are supposed to answer. When considered this way we can think of the Kolmogorov interpretation in terms of questions. For example, $a \vee b$ might be "What is the answer to either a or b?" Then axioms such as $a \to a \wedge a$ would represent trivially answerable questions such as "Given the answer to a what is the answer to both a and a?" This offers a connection between constructive proof and the semantics of questions. Second, Dummett's *The*

[1] For the quotation from Gentzen, see the beginning of Section 4.2.

[2] For the formal definition of PTV see Definition 4.8 and for the formal definition of quasi-PTV see Definition 6.5.

Logical Basis of Metaphysics is a defence of Prawitz's ideas, and a key component of Dummett's philosophy of language was that knowing a sentence's meaning involved knowing how to use the sentence (Dummett 1991, p. 103). Dummett thought of this as involving the ability to recognise when a statement has been verified. One way this might be spelt out is by identifying a verification with an answer to a query about the truth of the statement. More detailed consideration would take up beyond the scope of this paper, but this hopefully dissipates the concern that the results here are mere mathematical accidents.

This paper is organised as follows. Since neither inquisitive logic nor PTV are closed under uniform substitution, in Section 2 we discuss weak or non-structural logics which are not necessarily closed under substitution. Inquisitive logic and general inquisitive logic are discussed in more detail in Section 3. We define PTV both as Prawitz originally did on derivations and as a consequence relation and discuss the relationship between the two in Section 4. In Section 5 we discuss how to modify PTV by changing how one deals with atomic formulas. We prove several results about how modified systems relate to PTV and each other. We define quasi-PTV in Section 6. And it is there that our main theorems are proven and the consequences of these results for PTV are drawn out.

2 Logics Without Closure Under Substitution

Before we explore PTV and inquisitive semantics, we need to modify the definition of logics to include logics which aren't closed under substitution. Logics which aren't closed under substitution are called weak or non-structural (we will use weak). After defining weak logics, we will focus in on those weak logics which have the disjunction property (Definition 2.4). The paper then considers characterisations of weak logics found in Ciardelli and Roelofsen (2011) and Punčochář (2016). In later sections, this will allow us to show the extensional identity of quasi-PTV and inquisitive semantics and of PTV and general inquisitive semantics.

2.1 Weak Logics

The set of propositional formulas is built from the set of atomic formulas $\{p_i \mid i \in \mathbb{N}\} \cup \{\bot\}$ and the recursive application of the connectives $\wedge, \vee, \rightarrow$. Let a *logic* L be a subset of the propositional formulas closed under modus ponens and substitution. Note that we take $\bot$ to be an atomic proposition. Let a *weak logic* L be a set of propositional formulas closed under modus ponens, which need not be closed under substitution.

We define (weak) intermediate logics as follows:

Definition 2.1. A (weak) logic L is a *(weak) intermediate logic* if $IPC \subseteq L \subseteq CPC$.

Here we take IPC to be the deductive closure of the axioms of the intuitionistic propositional calculus, and similarly, CPC is the deductive closure of the axioms of the classical propositional calculus. While weak intermediate logics aren't necessarily closed under substitution, because they have IPC as a sublogic, they contain all substitution instances of the IPC axioms.

Definition 2.2. Let L be a (weak) intermediate logic then the consequence relation $\vdash_L$ is defined as follows:

$$\vdash_L \varphi \Leftrightarrow \varphi \in L$$

$$\Gamma \vdash_L \varphi \Leftrightarrow \text{ there are } \psi_1, \ldots, \psi_n \in \Gamma, (\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow \varphi \in L$$

Notice that we have defined the consequence relation so that the deduction theorem holds. The consequence relation of a weak intermediate logic also has the following desirable properties:

Lemma 2.3. Let L be a (weak) intermediate logic.

$$\nvdash_L \bot, \qquad \text{(Falsum Property)}$$
$$\vdash_L \varphi \wedge \psi \Longleftrightarrow \vdash_L \varphi \text{ and } \vdash_L \psi, \qquad \text{(Conjunction Property)}$$
$$\vdash_L \psi \to \varphi \Longleftrightarrow \psi \vdash_L \varphi, \qquad \text{(Weak Deduction Theorem)}$$
$$\varphi \vdash_L \varphi, \qquad \text{(Reflexivity)}$$
$$\Gamma \vdash_L \varphi \text{ and } \varphi \vdash_L \psi \Longrightarrow \Gamma \vdash_L \psi, \qquad \text{(Transitivity)}$$
$$\Gamma \vdash_L \varphi \Longleftrightarrow \exists \varphi_0, \ldots, \varphi_n \in \Gamma,\ \varphi_0, \ldots, \varphi_n \vdash_L \varphi. \qquad \text{(Compactness)}$$

Above we have a condition for $\bot$, $\wedge$, and $\to$ but no condition for $\vee$. The natural one is the following:

Definition 2.4. A logic L has the *disjunction property* if

$$\vdash_L \varphi \vee \psi \quad \Longleftrightarrow \quad \vdash_L \varphi \text{ or } \vdash_L \psi.$$

However, while this property holds in IPC, it does not hold in CPC and there are weak intermediate logics both with and without it.

2.2 Equality Between Weak Logics with the Disjunction Property

From now on we will work with logics which have the disjunction property. If a formula is a disjunction of negated formulas (i.e. $\neg\varphi_1 \vee \cdots \vee \neg\varphi_n$) then we say it is in *disjunctive negation form.* If it is a disjunction of disjunction free formulas (i.e. $\varphi_1 \vee \cdots \vee \varphi_n$ with φ_i not containing disjunction for all i) then we say it is in *disjunctive form.* We will see that two weak logics with the disjunction property are equal if both logics agree that every formula is equivalent to one in disjunctive negation form. And similarly, two weak logics with the disjunction property are equal if they agree on the disjunction free formulas and that every formula is equivalent to one in disjunctive form. We will give a translation into disjunctive and disjunctive negation form below in Definition 2.6 and 2.7. But first we will generalise results of Ciardelli and Roelofsen (2011, Theorem 3.2.36) and Punčochář (2016, Theorem 4).

Let $DF(L)$ be all disjunction free formulas of L. That is those formulas not containing disjunction.

Theorem 2.5. Suppose L_1, L_2 are weak intermediate logics such that they both have the disjunction property. Assume that both logics satisfy the same condition, either:

(2.5.1.) for all φ there is a ψ in *disjunctive negation form* such that $\psi \equiv \varphi \in L_1, L_2$, or

(2.5.2.) for all φ there is a ψ in *disjunctive form* such that $\psi \equiv \varphi \in L_1, L_2$ and $DF(L_1) = DF(L_2)$.

Then $L_1 = L_2$.

Proof. We prove the first case first. Without loss of generality assume $\varphi \in L_1$. Then there is a ψ in disjunctive negation form such that $\psi \in L_1$. Let ψ be $\neg\varphi_0 \vee \cdots \vee \neg\varphi_n$. As L_1 has the disjunction property there is an $i \leq n$ such that $\neg\varphi_i \in L_1$. Because L_1 is a sublogic of CPC it follows that

$\neg\varphi_i$ is a tautology of classical logic and so by Glivenko's theorem $\neg\neg\neg\varphi_i$ is a tautology of IPC and so $\neg\varphi_i$ is a tautology of IPC. It follows then that $\neg\varphi_0 \vee \cdots \vee \neg\varphi_n$ is a tautology of IPC by disjunction introduction. And as IPC is a sublogic of L_2 it follows that $\neg\varphi_0 \vee \cdots \vee \neg\varphi_n \in L_2$ and so $\varphi \in L_2$ as $\psi \equiv \varphi \in L_2$. So L_1 is a sublogic of L_2. And as the same reasoning goes through with L_1 switched with L_2 it follows that the two logics must be equal.

The proof of the second is very similar. The difference is that one assumes ψ is $\varphi_0 \vee \cdots \vee \varphi_n$ where φ_i is disjunction free for all i. And, rather than going through IPC, one uses the assumption that $DF(L_1) = DF(L_2)$ to get that there is a $\varphi_i \in L_2$ for some i. □

This theorem gives us two ways to characterise weak intermediate logics with the disjunction property.

We now want to show when a logic is such that every formula is equivalent to one in disjunctive or disjunctive negation form. It will be helpful to have a particular translation of formulas into disjunctive and disjunctive negation form. The following definition of a disjunctive negation translation is a variation on one by Maksimova (1986) employed by Ciardelli and Roelofsen (2011). The definition of disjunctive translation is the obvious modification.

Definition 2.6. Let the disjunctive translation DT be as follows:

$$DT(p) = p \tag{1}$$

$$DT(\varphi \vee \psi) = DT(\varphi) \vee DT(\psi) \tag{2}$$

$$DT(\varphi \wedge \psi) = \bigvee\{\varphi_i \wedge \psi_j \mid 0 < i \leq n, 0 < j \leq m\} \tag{3}$$

$$\text{where } DT(\varphi) = \varphi_1 \vee \cdots \vee \varphi_n \text{ and } DT(\psi) = \psi_1 \vee \cdots \vee \psi_m$$

$$DT(\varphi \to \psi) = \bigvee\{\bigwedge_{0<j\leq n} (\varphi_j \to \psi_{i_j}) \mid (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n\} \tag{4}$$

$$\text{where } DT(\varphi) = \varphi_1 \vee \cdots \vee \varphi_n \text{ and } DT(\psi) = \psi_1 \vee \cdots \vee \psi_m$$

Definition 2.7. Let the disjunctive negation translation DNT be as follows:

$$DNT(p) = \neg\neg p \tag{5}$$

$$DNT(\varphi \vee \psi) = DNT(\varphi) \vee DNT(\psi) \tag{6}$$

$$DNT(\varphi \wedge \psi) = \bigvee\{\neg(\varphi_i \vee \psi_j) \mid 0 < i \leq n, 0 < j \leq m\} \tag{7}$$

$$\text{where } DNT(\varphi) = \neg\varphi_1 \vee \cdots \vee \neg\varphi_n \text{ and } DNT(\psi) = \neg\psi_1 \vee \cdots \vee \neg\psi_m$$

$$DNT(\varphi \to \psi) = \bigvee\{\neg\neg \bigwedge_{0<j\leq n} (\psi_{i_j} \to \varphi_j) \mid (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n\} \tag{8}$$

$$\text{where } DNT(\varphi) = \neg\varphi_1 \vee \cdots \vee \neg\varphi_n \text{ and } DNT(\psi) = \neg\psi_1 \vee \cdots \vee \neg\psi_m$$

The definition of DNT looks confusing, particularly for $\wedge$ and $\to$, but some examples will hopefully illuminate the translation. The translation of $\varphi \wedge \psi$ states that there is some φ_i and ψ_i such that $\neg(\varphi_i \vee \psi_i)$ is true. So $(\neg\varphi_1 \vee \neg\varphi_2) \wedge (\neg\psi_1 \vee \neg\psi_2)$ becomes

$$\neg(\varphi_1 \vee \psi_1) \vee \neg(\varphi_1 \vee \psi_2) \vee \neg(\varphi_2 \vee \psi_1) \vee \neg(\varphi_2 \vee \psi_2).$$

The DNT translation of $\varphi \to \psi$ is the disjunction of all the ways the φ_i's could be implied by some of the ψ_j's with a double negation on the front. For example, the DNT translation of

$(\neg\varphi_1 \vee \neg\varphi_2) \to (\neg\psi_1 \vee \neg\psi_2)$ is

$$\neg\neg((\psi_1 \to \varphi_1) \wedge (\psi_1 \to \varphi_2)) \vee \neg\neg((\psi_1 \to \varphi_1) \wedge (\psi_2 \to \varphi_2))$$
$$\vee\neg\neg((\psi_2 \to \varphi_1) \wedge (\psi_1 \to \varphi_2)) \vee \neg\neg((\psi_2 \to \varphi_1) \wedge (\psi_2 \to \varphi_2))$$

Now that we have translations in mind, we can think about what properties a logic would have to have to prove $\varphi \leftrightarrow DT(\varphi)$ or $\varphi \leftrightarrow DNT(\varphi)$. We are interested in what properties it would have to have on top of IPC. We need $\to$ to commute over $\vee$, for disjunction free antecedents in the case of DT and for negated antecedents in the case of DNT. In the case of negated antecedents this condition is the widely studied Kreisel-Putnam axiom[3]:

Definition 2.8.

$$(\neg\varphi \to \psi \vee \chi) \to [(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)] \tag{KP}$$

For disjunction free antecedents the natural generalisation of the Kreisel-Putnam axiom is needed:

Definition 2.9. For all disjunction free φ,

$$(\varphi \to \psi \vee \chi) \to [(\varphi \to \psi) \vee (\varphi \to \chi)] \tag{GKP}$$

If one simply adds the Kreisel-Putnam axiom to IPC the result is Kreisel-Putnam logic KP. If one adds the generalised Kreisel-Putnam axiom to intuitionistic logic then generalised Kreisel-Putnam logic or $IPC + GKP$ is the result.[4] Note that in IPC, $\neg\varphi$ is equivalent to some formula φ^* not containing disjunction (Kleene 1952, Sec. 26-7) so the generalised Kreisel-Putnam axiom implies the Kreisel-Putnam axiom.

To prove that $DNT(\varphi)$ is equivalent to φ we also need the equivalence of p and $\neg\neg p$. This is not a theorem of IPC as, while $IPC \vdash p \to \neg\neg p$, famously $IPC \nvdash \neg\neg p \to p$. So, we will need a logic which has $\neg\neg p \to p$ for all atomic formulas. In fact, in the presence of double negation elimination for atomic formulas, KP implies GKP (Punčochář 2016).[5] From these consideration we can conclude that any logic extending $IPC+GKP$ proves the equivalence of every formula with its DT translation and if it also contains double negation elimination for atomic formulas, it proves the equivalence of every formula with its DNT translation.

Lemma 2.10. If L is a weak intermediate logic with the disjunction property and the generalised Kreisel-Putnam axiom then:

(2.10.1.) for all φ, we have that $\vdash_L \varphi \leftrightarrow DT(\varphi)$,

(2.10.2.) if $\vdash_L \neg\neg p \to p$ for all atomic p, then for all φ, we have $\vdash_L \varphi \leftrightarrow DNT(\varphi)$.

[3]The inference rule that goes from the antecedent to the consequence of the Kreisel-Putnam axiom is called Harrop's rule. And it is this name that is often used in work on proof-theoretic validity.

[4]The following three things are worth noting. First, $IPC + GKP$ is equivalent to one where GKP restricts φ to the Harrop formulas (roughly a formula where disjunction may occur only in the antecedent of an implication). In this second guise it has been studied by Punčochář 2016 and Miglioli et al. 1989. Second, even though we are working with weak logics, we will only be concerned with logics which contain every substitution instance of the standard or generalised Kreisel-Putnam axiom. Third, the converse of both the generalised and standard Kreisel-Putnam axiom is provable in IPC, so we can replace the main $\to$ with a $\leftrightarrow$.

[5]Using DNT it is easy to see (via Lemma 2.10) that every disjunction free formula is equivalent to one starting with a negation *if* atomic formulas are.

Proof. First, we prove the equivalence of φ with $DT(\varphi)$. The base case is trivial, as is the disjunction case. The conjunctive case simply uses the distributive properties of conjunction and disjunction provable in IPC.

For the case of implication, note that $\varphi_1 \vee \cdots \vee \varphi_n \to \psi$ is equivalent to $(\varphi_1 \to \psi) \wedge \cdots \wedge (\varphi_n \to \psi)$ and as φ_i for all i is disjunction free, $\varphi_i \to \psi_1 \vee \cdots \vee \psi_m$ is equivalent to $(\varphi_i \to \psi_1) \vee \cdots \vee (\varphi_i \to \psi_m)$ by application of GKP. This means that $\varphi_1 \vee \cdots \vee \varphi_n \to \psi_1 \vee \cdots \vee \psi_m$ is equivalent to $[(\varphi_1 \to \psi_1) \vee \cdots \vee (\varphi_1 \to \psi_m)] \wedge \cdots \wedge [(\varphi_n \to \psi_1) \vee \cdots \vee (\varphi_n \to \psi_m)]$. And by the distributive properties of IPC this is equivalent to $\bigvee\{\bigwedge_{0<j\leq n}(\varphi_j \to \psi_{i_j}) \mid (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n\}$.

We turn now to the second case. Note that by hypothesis $\vdash_L \neg\neg p \to p$ so $DNT(p) \equiv_L p$. The disjunction case is trivial. To show the conjunction case in the right to left direction holds in IPC we first assume $\bigvee\{\neg(\varphi_i \vee \psi_j) \mid 0 < i \leq n, 0 < j \leq m\}$ and then apply $\vee$ elimination. Note that $\neg(\varphi_i \vee \psi_j)$ is equivalent in IPC to $(\neg\varphi_i \wedge \neg\psi_j)$. So, by disjunction introduction $\neg\varphi_1 \vee \cdots \vee \neg\varphi_n \vee \neg\psi_1 \vee \cdots \vee \neg\psi_m$ follows. From which it follows by the induction hypothesis that $\vdash_L \varphi \wedge \psi$.

For the other direction assume $\varphi \wedge \psi$. By the induction hypothesis $(\neg\varphi_1 \vee \cdots \vee \neg\varphi_n) \wedge (\neg\psi_1 \vee \cdots \vee \neg\psi_m)$. So, by using the distribution properties in IPC we get $\bigvee\{(\neg\varphi_i \wedge \neg\psi_j) \mid 0 < i \leq n, 0 < j \leq m\}$. But $\neg\varphi_i \wedge \neg\psi_j$ is equivalent to $\neg(\varphi_i \vee \psi_j)$ in IPC. So, we get we get $\bigvee\{\neg(\varphi_i \vee \psi_j) \mid 0 < i \leq n, 0 < j \leq m\}$.

This leaves $\to$. This follows from the induction hypothesis and the following derivation in GKP.

$$\vdash_{GKP} (\neg\varphi_1 \vee \cdots \vee \neg\varphi_n) \to (\neg\psi_1 \vee \cdots \vee \neg\psi_m)$$

if and only if (by the DT translation):

$$\vdash_{GKP} \bigvee\{\bigwedge_{0<j\leq n}(\neg\varphi_j \to \neg\psi_{i_j}) \mid (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n\}$$

if and only if (because in IPC we have $(\neg A \to \neg B) \leftrightarrow (\neg\neg B \to \neg\neg A)$)

$$\vdash_{GKP} \bigvee\{\bigwedge_{0<j\leq n}(\neg\neg\psi_{i_j} \to \neg\neg\varphi_j) \mid (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n\}$$

if and only if (because in IPC double negation commutes over implication and conjunction)

$$\vdash_{GKP} \bigvee\{\neg\neg\bigwedge_{0<j\leq n}(\psi_{i_j} \to \varphi_j) \mid (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n\}. \qquad \square$$

By previous remarks about the relation between KP and GKP, it follows from this Lemma that $\vdash_L \varphi \leftrightarrow DNT(\varphi)$ holds for any weak intermediate logic L with the disjunction property, double negation elimination for atomic formulas and *standard* the Kreisel-Putnam axiom. We can now generalise Theorem 2.5.

Corollary 2.11. Let L_1, L_2 be two weak intermediate logics such that one of the following conditions is satisfied by both logics:

(2.11.1.) disjunction property and all formulas equivalent to their DNT translation,

(2.11.2.) disjunction property, the Kreisel-Putnam axiom and $\neg\neg p \to p$,

(2.11.3.) disjunction property, the generalised Kreisel-Putnam axiom and $\neg\neg p \to p$.

It follows that $L_1 = L_2$.

Corollary 2.12. Further, let L_1, L_2 be two weak intermediate logics such that one of the following conditions is satisfied by both logics:

(2.12.1.) disjunction property, $DF(L_1) = DF(L_2)$, and all formulas equivalent to their DT translation,

(2.12.2.) disjunction property, $DF(L_1) = DF(L_2)$, and the generalised Kreisel-Putnam axiom.

It follows that $L_1 = L_2$.

3 Inquisitive Logic

Inquisitive semantics is a formal semantics designed to offer a uniform treatment of assertions and questions. This is motivated by observations such as the mutual embedding of sentences and questions. For example:

Xiao asked if Anna is here.	(embedded question)	(9)
Who told you that Anna is here?	(embedded assertion)	(10)
Xiao asked me who told you that Anna is here.	(two-level embedding)	(11)

Further motivations include the use of logical connectives in both questions and assertions, that answers to questions are interpreted with contextual information given in the question, and that propositional attitudes can have questions as their objects (Ciardelli, Groenendijk, and Roelofsen 2018, Ch. 1).

Traditionally an assertion has been modelled by a set of possible worlds (Stalnaker 1976) while a question is treated as the set of possible answers to the question (Karttunen and Peters 1980; Groenendijk and Stokhof 1984). As answers to questions are assertions, this means that a question is a set of assertions. So, questions can be treated as sets of sets of possible worlds. These different treatments rule out a uniform treatment of "Is Anna here?" and "Anna is here." The solution proposed by inquisitive semantics is to treat both assertions and questions in the spirit of the traditional treatment of questions. This is done by treating propositions as sets of sets of possible worlds closed under subsets.

To make this precise, first, we need a collection of possible worlds W and using this we can define an information state.

Definition 3.1. An *information state* $s \subseteq W$ is a set of possible worlds.

Note that in the more traditional setting an information state would be a proposition. In that setting, you can think of a proposition as having more information the fewer worlds it contains. This is because you can think of a proposition as containing all the information the worlds have in common and fewer worlds mean more shared information. It follows that if a proposition p is a subset of another proposition q, then p has more information than q. This justifies calling a set of worlds that is a subset of another an enhancement of it.

Definition 3.2. A state t is an *enhancement* of s if $t \subseteq s$.

In the setting of inquisitive logic, a proposition is then defined as a set of information states closed under enhancements.

Definition 3.3. A *proposition* is a non-empty set P of information states which is closed downwards under enhancements.

A proposition must be non-empty because containing no information is associated with the proposition that is always false, which contains one set namely the empty one. Being closed under enhancements means that an inquisitive proposition is a set which contains every traditional proposition which implies any other traditional proposition in the set.

A proposition in inquisitive logic is an assertion if its union is a member of it. What this amounts to is that it can be treated as a set of possible worlds because it has a least informative set in it. This means that if P is a set of possible worlds, $\mathcal{P}(P) = \{Q : Q \subseteq P\}$ is its translation into inquisitive semantics. Propositions that don't have this property represent questions (here they are called *inquisitive*). Inquisitive propositions contain at least two distinct maximal sets of worlds which can be thought of as representing distinct answers to a question. Formally:

Definition 3.4. P is *inquisitive* iff $\bigcup P \notin P$.

The language of propositional logic can be interpreted in this setup.

Definition 3.5. Let $V : At \to \mathcal{P}(W)$ be a valuation function on atomic propositions, then define $[\ \cdot\ ]_{\langle W,V\rangle} : Prop \to \mathcal{P}(\mathcal{P}(W))$ as a function from the language of propositional logic to sets of information states meeting the following conditions:

$$[p]_{\langle W,V\rangle} = \{s \mid s \subseteq V(p)\} \tag{12}$$

$$[\varphi \wedge \psi]_{\langle W,V\rangle} = [\varphi]_{\langle W,V\rangle} \cap [\psi]_{\langle W,V\rangle} \tag{13}$$

$$[\varphi \vee \psi]_{\langle W,V\rangle} = [\varphi]_{\langle W,V\rangle} \cup [\psi]_{\langle W,V\rangle} \tag{14}$$

$$[\varphi \to \psi]_{\langle W,V\rangle} = \{s \mid \forall t \subseteq s\ (t \in [\varphi]_{\langle W,V\rangle} \Rightarrow t \in [\psi]_{\langle W,V\rangle})\} \tag{15}$$

Here $\bot$ is not true at any world, so that $[\bot]_{\langle W,V\rangle} = \{\emptyset\}$. We can prove by an easy induction that for all φ the set $[\varphi]_{\langle W,V\rangle}$ will be a proposition.[6] When W and V are clear from context, they will be omitted and we will write $[\varphi]$ instead of $[\varphi]_{\langle W,V\rangle}$.

With $[\cdot]_{\langle W,V\rangle}$ defined we can define inquisitive logic as follows:

Definition 3.6. $\varphi \in L_{Inq}$ if and only if for all W, V one has $[\varphi]_{\langle W,V\rangle} = \mathcal{P}(W)$.

A cursory examination will show only formulas containing $\vee$ are inquisitive. What explanation is there for this identification of disjunctions with questions? Ciardelli, Groenendijk, and Roelofsen (2018, pp. 73–4) appeal to cross-linguistic evidence that the same 'words' are used for questions and disjunctions. For example, in Japanese the particle 'ka' is used at the end of a sentence to signal a question ('Anna wa kita-**ka**' Did Anna come?) and attached to each noun to signal a disjunction ('Anna-**ka** Xiao-**ka**' Anna or Xiao). It has been proposed that inquisitive semantics can account for this data (Szabolesi 2015). Reflection on the role of disjunction might also suggest a relation to questions. A disjunction involves in some sense a loss or lack of information–you do not know which of the two disjuncts are true. Questions similarly involve a lack of information. While this

[6]It is also worth noting that the definition of $\to$ is equivalent to the more traditional definition of $sup\{[\chi]_{\langle W,V\rangle} \mid [\chi \wedge \varphi]_{\langle W,V\rangle} \subseteq [\psi]_{\langle W,V\rangle}\}$ from Heyting algebras (Troelstra and Dalen 1988, ch. 13).

might be thought to be a pragmatic feature of disjunction, inquisitive semantics accounts for it in the semantics.

It also turns out that negated formulas are uninquisitive.

Lemma. Given a W, for all φ we have that $\bigcup[\neg\varphi] \in [\neg\varphi]$. In other words, $\neg\varphi$ is uninquisitive.

Proof. We will show that $\bigcup\{s \mid \forall t \in [\varphi]\ t \cap s = \varnothing\} \in \{s \mid \forall t \in [\varphi]\ t \cap s = \varnothing\}$. Assume otherwise. Then there would be a $t \in [\varphi]$ such that $t \cap \bigcup\{s \mid \forall t \in [\varphi]\ t \cap s = \varnothing\} \neq \varnothing$ but then there would have to be a $s \in \{s \mid \forall t \in [\varphi]\ t \cap s = \varnothing\}$ such that $t \cap s \neq \varnothing$ but this is a contradiction. □

It follows that inquisitive logic has double negation elimination for atomic formulas.

Lemma 3.7 (Ciardelli and Roelofsen 2011, Rmk 3.8, p. 10)**.** $\neg\neg p \to p$ holds in inquisitive logic

Proof. Atomic propositions aren't inquisitive, and this result holds for all uninquisitive formulas. It follows from the fact that φ is not inquisitive then $[\neg\neg\varphi] = [\varphi]$, which we will now prove. Note that

$$[\neg\neg\varphi] = \{s \mid \forall t[(\forall v \in [\varphi](t \cap v = \varnothing)) \Rightarrow t \cap s = \varnothing]\}. \tag{16}$$

Clearly $[\varphi] \subseteq [\neg\neg\varphi]$. To show the other direction we will first show that $\bigcup[\neg\neg\varphi] = \bigcup[\varphi]$. Assume for a contradiction that $w \in \bigcup[\neg\neg\varphi]$ and assume $w \notin \bigcup[\varphi]$. It follows that for all $v \in [\varphi]$ we have $\{w\} \cap v = \varnothing$ but $\{w\} = \{w\} \cap \bigcup[\neg\neg\varphi]$. From this it follows by equation 16 that $\bigcup[\neg\neg\varphi] \notin [\neg\neg\varphi]$. But $[\neg\neg\varphi]$ isn't inquisitive by the Lemma above and so contains $\bigcup[\neg\neg\varphi]$. Therefore, this is a contradiction. As φ isn't inquisitive, it has $\bigcup[\varphi]$ as its maximal element. We just proved that the maximal elements of both sets are the same. And so, since they are both closed downwards, they are the same set. □

Note that inquisitive logic is a weak logic as $\neg\neg\varphi \to \varphi$ does not hold for inquisitive propositions. For example, $[\neg\neg(\varphi \vee \psi)]$ is the downwards closure of $\bigcup[\varphi \vee \psi]$ while $[\varphi \vee \psi] = [\varphi] \cup [\psi]$. Inquisitive logic also satisfies the axioms of IPC (Ciardelli and Roelofsen 2011, Prop 3.19, p. 14), the disjunction property (Ciardelli and Roelofsen 2011, Prop 3.9, p. 10), and the Kreisel-Putnam axiom (Ciardelli and Roelofsen 2011, Rmk 3.8, p. 10). As such it is characterised by the second condition of Corollary 2.11.

There is another way of defining inquisitive semantics by defining a support relation between states and formulas. This definition will allow the construction of a more generalised version of inquisitive semantics. To do this we will need to define inquisitive systems.

Definition 3.8. Given a set of worlds W, an $I \subseteq \mathcal{P}(W)$, a valuation function $V : At \to I$, the triple $\langle W, I, V\rangle$ is an *inquisitive system.*

We can then give the following definition as found in Punčochář 2016.

Definition 3.9. Given an inquisitive system $\langle W, I, V\rangle$ and a state s in I, s supports φ in $\langle W, I, V\rangle$ or $s \vDash^{Inq}_{\langle W,I,V\rangle} \varphi$ when the following conditions are met:

(3.9.1.) $s \vDash^{Inq}_{\langle W,I,V\rangle} p$ iff $s \subseteq V(p)$,

(3.9.2.) $s \vDash^{Inq}_{\langle W,I,V\rangle} \bot$ iff $s = \varnothing$,

(3.9.3.) $s \vDash^{Inq}_{\langle W,I,V\rangle} \varphi \wedge \psi$ iff $s \vDash^{Inq}_{\langle W,I,V\rangle} \varphi$ and $s \vDash^{Inq}_{\langle W,I,V\rangle} \psi$, (3.9.4.) $s \vDash^{Inq}_{\langle W,I,V\rangle} \varphi \vee \psi$ iff $s \vDash^{Inq}_{\langle W,I,V\rangle} \varphi$ or $s \vDash^{Inq}_{\langle W,I,V\rangle} \psi$,

(3.9.5.) $s \vDash^{Inq}_{\langle W,I,V\rangle} \varphi \rightarrow \psi$ iff for all $t \subseteq s$ if $t \in I$ and $t \vDash^{Inq}_{\langle W,I,V\rangle} \varphi$ then $t \vDash^{Inq}_{\langle W,I,V\rangle} \psi$.

We will write $L_{\langle W,I,V\rangle}$ for the set of formulas supported by every $s \in I$ in an inquisitive system $\langle W, I, V\rangle$. As pointed out by Punčochář (2016, p. 410), this new definition connects to Definition 3.5 via the following result:

Proposition 3.10. Given $s \in \mathcal{P}(W)$ it follows that $s \vDash^{Inq}_{\langle W,\mathcal{P}(W),V\rangle} \varphi$ if and only if $s \in [\varphi]_{\langle W,V\rangle}$. So $\varphi \in L_{Inq}$ if and only if for all W and V and all states $s \in \mathcal{P}(W)$, we have $s \vDash^{Inq}_{\langle W,\mathcal{P}(W),V\rangle} \varphi$. Which is to say, $\varphi \in L_{Inq}$ if and only if for all W and V, we have $\varphi \in L_{\langle W,\mathcal{P}(W),V\rangle}$

Let *general inquisitive semantics* be the set of all inquisitive systems $\langle W, I, V\rangle$ where I is a topology on W. So, I is closed under finite intersections and arbitrary unions. Let L_{GInq} or *general inquisitive logic* be thevery formula which is supported by every state in every inquisitive system in general inquisitive semantics. In other words, L_{GInq} is the logic of general inquisitive semantics.

It can be shown that general inquisitive logic shares several important features with inquisitive logic. Like inquisitive logic, it is known that general inquisitive logic satisfies IPC. This result follows from the below equivalence with Kripke models:

Lemma 3.11 (Punčochář 2016, p. 412)**.** Given $\langle W, I, V\rangle$, the Kripke semantics $\langle I - \{\varnothing\}, \supseteq, V^*\rangle$ where $V^*(p) = \{s \in I - \{\varnothing\} \mid s \subseteq V(p)\}$ defines the same satisfaction relation.

We can also show that general inquisitive logic has the disjunction property just like inquisitive logic. This is a generalisation of (Ciardelli and Roelofsen 2011, Prop 3.9, p. 10).

Lemma 3.12. If I is a topology on W, then $L_{\langle W,I,V\rangle}$ has the disjunction property.

Proof. Assume $\varphi \vee \psi \in L_{\langle W,I,V\rangle}$. As I is a topology there is $t \in I$ that for all $s \in I$, it follows that $s \subseteq t$. It follows that $t \vDash^{Inq}_{\langle W,I,V\rangle} \varphi$ or $t \vDash^{Inq}_{\langle W,I,V\rangle} \chi$. As propositions are downwards closed it follows that for all $s \in I$ either $s \vDash^{Inq}_{\langle W,I,V\rangle} \varphi$ or $s \vDash^{Inq}_{\langle W,I,V\rangle} \psi$. So $\varphi \in L_{\langle W,I,V\rangle}$ or $\psi \in L_{\langle W,I,V\rangle}$. □

Lemma 3.13 (Punčochář 2016, Lem. 5)**.** L_{GInq} has the disjunction property.

(Sketch). Assume not, then there is a formula $\varphi \vee \psi \in L_{GInq}$ and there are two models $\langle W, I, V\rangle$ and $\langle W', I', V'\rangle$ such that $\varphi \notin L_{\langle W,I,V\rangle}$ and $\psi \notin L_{\langle W',I',V'\rangle}$. But given this, we would be able to construct a model from the disjoint union of the set of worlds W and W' such that the new model doesn't prove either disjunct. □

And like inquisitive logic, general inquisitive logic contains the generalised Kreisel-Putnam axiom:

Lemma 3.14 (Punčochář 2016, p. 420)**.** Every instance of the generalised Kreisel-Putnam axiom holds in every $\langle W, I, V\rangle$ where I is a topology on W.

Proof. Note that if φ is disjunction free then it is uninquisitive so with a topological I there is a maximal information state s such that $s \vDash^{Inq}_{\langle W,I,V\rangle} \varphi$. Let $s \in I$ be such that $s \vDash^{Inq}_{\langle W,I,V\rangle} \varphi \rightarrow \psi \vee \chi$ where φ is disjunction free. Assume for a contradiction that $s \nvDash^{Inq}_{\langle W,I,V\rangle} (\varphi \rightarrow \psi) \vee (\varphi \rightarrow \chi)$. It then follows that there are $t, r \in I$ which are subsets of s, and t supports φ and ψ but not χ, and r supports φ and χ but not ψ. As I is topological $t \cup r \in I$ and because φ is uninquisitive it follows that $t \cup r$ supports φ. Inspection of Definition 3.9 shows that if a state supports a proposition then all its substates will do so to. Hence $t \cup r$ supports $\varphi \to \psi \vee \chi$ and so it either supports ψ or χ. But then so must t and r which contradicts our assumption. □

General inquisitive logic differs from inquisitive logic when it comes to double negation elimination for atomic formulas. This is not a theorem of general inquisitive logic. In fact, it follows from Corollary 5 of Punčochář 2016 that L_{GInq} is the same logic as $IPC + GKP$, a sublogic of L_{Inq}. And from Theorem 4 of Punčochář 2016 it follows that the disjunction free fragment of L_{GInq} is the same as the disjunction free fragment of intuitionistic logic. We will use this result later along with Corollary 2.12 to show that the logic of PTV is general inquisitive logic.

4 Proof-Theoretic Validity

Thus far we have been discussing weak logics and inquisitive logic. We now turn to proof-theoretic semantics. We begin in this section by introducing the notion of an atomic system and a proof-theoretic system. Then we will define two notions of proof-theoretic validity relative to a proof-theoretic system. The first definition is due to Prawitz (1973). The second is due to Piecha, Campos Sanz, and Schroeder-Heister (2015). We will show that these two definitions aren't equivalent (see Lemma 4.16).

4.1 Atomic Systems

To define proof-theoretic validity we first need to define a very general notion of a proof rule for an atomic formula:

Definition 4.1. (Schroeder-Heister 1984) We define atomic rules and their levels as follows:

(4.1.1.) A *level-0 atomic rule* is an axiom consisting of a single atomic formula. That is a rule with no premises or hypotheses. It is written as $/p$,

(4.1.2.) A *level-1 atomic rule* is a rule which has premises but does not discharge hypotheses. Written $p_0, \ldots, p_n/q$ for an inference from $p_0, \ldots, p_n$ to q,

(4.1.3.) A *level-2 atomic rule* is a rule which discharge hypotheses. Written

$$[p_{0_0}, \ldots, p_{m_0}]q_0, \ldots, [p_{0_n}, \ldots, p_{m_n}]q_n/r$$

for an inference from $q_0, \ldots, q_n$ to r, which for each q_i discharges $p_{0_i}, \ldots, p_{m_i}$,

(4.1.4.) A *level-n atomic rule* (for $n > 2$) is a rule which discharge rules of level-$(n-2)$. Written

$$([R_{0_0}, \ldots, R_{m_0}]q_0), \ldots, ([R_{0_n}, \ldots, R_{m_n}]q_n)/r$$

for a level-n inference from $q_0, \ldots, q_n$ to r, which for each q_i discharges rules $R_{0_i}, \ldots, R_{m_i}$, where the level of each R_j is at most level-$(n-2)$.

We will call any set S of atomic rules an *atomic system*. This definition can seem confusing, particularly for rules above 2. To help with this we will first give some examples of rules and then proofs containing those rules.

Example 4.2. Consider the following examples:

$$\text{level 0: } \overline{p} \qquad \text{level 1: } \frac{p}{q} \qquad \text{level 2: } \begin{array}{c}[p]\\ \vdots \\ \dfrac{q}{r}\end{array} \qquad \text{level 3: } \begin{array}{c}\vdots \\ \left[\dfrac{p}{q}\right] \\ \vdots \\ \dfrac{q}{r}\end{array}$$

Using these rules we can demonstrate how to construct proofs that discharge rules. To illustrate the discharge of rules it will help to write each step φ, D where φ is atomic and D is the set of atomic rules used so far and not discharged.

Example 4.3. Consider the atomic system containing the level 3 rule above, written $[p/q]q/r$, and $/p$. We can then prove r as follows:

$$\dfrac{\dfrac{\overline{p, \{/p\}}}{q, \{/p, \mathbf{p/q}\}}}{r, \{/p, [\mathbf{p/q}]q/r\}}$$

As a second example, take the following proof in an atomic system with the level 5 rule $[/p, [p/q]q/r]s/t$ and the level 1 rule r/s. Note that the first three lines are identical to the proof above.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p, \{/p\}}}{q, \{/p, p/q\}}}{r, \{/p, [p/q]q/r\}}}{s, \{\mathbf{/p, [p/q]q/r}, r/s\}}}{t, \{[\mathbf{/p, [p/q]q/r}]s/t, r/s\}}$$

For a given atomic system S, we define $\vdash_S$ as a relation between a set of hypothesis and rules, and an atomic proposition. So $p_1, \ldots, p_n, R_1, \ldots, R_m \vdash_S p$ means there is a proof of p with open hypothesis $p_1, \ldots, p_n$ containing only the rules in $\{R_1, \ldots, R_m\} \cup S$. So, in the example above the proofs witness $\vdash_{\{[p/q]q/r,/p\}} r$ and $\vdash_{\{[[p/q]q/r,/p]s/t,r/s\}} t$. We will write $q \vdash_S p$ and $/q \vdash_S p$ interchangeably as there is no real effect to swapping axioms and atomic assumptions. Note the following observation:

Fact 4.4. Rules are only removed when a level 2 or higher rule is applied. It follows that a sub-proof of a proof with level-2 or higher rules in it may have new assumptions and new rules. But the sub-proofs of a proof with only level 0 and 1 rules contain the same rules as the initial proof (though they may have more assumptions).

This means that we can split proofs containing only level-0 or level-1 rules. First we give an illustrative example, and then we prove the general splitting result.

Example 4.5. Given the following proof of $p/q \vdash_{\{/s;s/p;q/r\}} r$:

$$\dfrac{\dfrac{\dfrac{\overline{\ s\ }}{p}}{q}}{r}$$

We can split the proof into $\dfrac{\overline{s}}{p}$ of $\vdash_{\{/s;s/p;q/r\}} p$ and $\dfrac{q}{r}$ of $/q \vdash_{\{/s;s/p;q/r\}} r$.

Lemma 4.6. If $\mathcal{D}$ is a proof in an atomic system S, from distinct rules $p/q, R_0, \ldots, R_n$ with conclusion r–that is if $\mathcal{D}$ witnesses $p/q, R_0, \ldots, R_n \vdash_S r$–and all the rules in S and $R_0, \ldots, R_n$ are level 0 or 1 rules, then if $\mathcal{D}$ contains p/q it follows that there are $\mathcal{D}_1$ and $\mathcal{D}_2$ witnessing $R_0, \ldots, R_n \vdash_S p$ and $q, R_0, \ldots, R_n \vdash_S r$, respectively.

Proof. The proof proceeds by induction on the number of instances of p/q in $\mathcal{D}$.

For the base case, assume there are no instances of p/q in $\mathcal{D}$ it follows that we are done simply because the antecedent is not satisfied.

Now let us assume for all $i < m$ that the induction hypothesis holds and that $\mathcal{D}$ contains m instances of p/q. Take any such occurrence and split $\mathcal{D}$ into two proofs $\mathcal{D}_1$ and $\mathcal{D}_2$ as follows:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{p}}{q}}{\dfrac{\mathcal{D}_2}{r}}$$

If $\mathcal{D}$ contained m occurrences of p/q then $\mathcal{D}_1$ and $\mathcal{D}_2$ contain i and j respectively where $m = i+j+1$ to account for the application of p/q dividing $\mathcal{D}_1$ and $\mathcal{D}_2$. There are four cases we need to deal with depending on whether or not $i, j > 0$. That is $m = 1$ and $i = j = 0$; $i = 0$ and $j = m-1$; $i = m-1$ and $j = 0$; or both $i, j > 0$. However, it is easier to consider the subcases $i > 0$, $j > 0$, $i = 0$, and $j = 0$ separately. When considering i we show how to get a proof witnessing $R_0, \ldots, R_n \vdash_S p$ and for j we show how to get a proof witnessing $q, R_0, \ldots, R_n \vdash_S r$. Then for the four cases above one simply combines the relevant pieces of the proof.

First we consider $i > 0$. We note by Fact 4.4 that $\mathcal{D}_1$ does not contain any rules $\mathcal{D}$ didn't. Further, because the top of the derivation remains unchanged, no new assumptions are added. This means it witnesses $p/q, R_0, \ldots, R_n \vdash_S p$. We then use the induction hypothesis to get two proofs: $\mathcal{D}_{1_1}$ witnessing $R_0, \ldots, R_n \vdash_S p$ and $\mathcal{D}_{1_2}$ witnessing $q, R_0, \ldots, R_n \vdash_S p$. The derivation $\mathcal{D}_{1_1}$ is the first proof we were looking for and we are done with $i > 0$.

Second we consider $j > 0$. We note by Fact 4.4 that $\mathcal{D}_2$ does not contain any rules $\mathcal{D}$ didn't, however it does have a new assumption q. So $\mathcal{D}_2$ witnesses $q, p/q, R_0, \ldots, R_n \vdash_S r$. So by the induction hypothesis we can split it into two proofs: $\mathcal{D}_{2_1}$ witnessing $q, R_0, \ldots, R_n \vdash_S p$ and $\mathcal{D}_{2_2}$ witnessing $q, R_0, \ldots, R_n \vdash_S r$. The derivation $\mathcal{D}_{2_2}$ is the second proof we were looking for and we are done with $j > 0$.

Third assume $i = 0$ then we still know $\mathcal{D}_1$ witnesses $p/q, R_0, \ldots, R_n \vdash_S p$ but we also know it contains no instances of p/q, so in fact it witnesses $R_0, \ldots, R_n \vdash_S p$. The derivation $\mathcal{D}_1$ is the third proof we were looking for and we are done with $i = 0$.

Fourth assume $j = 0$ then just as before we have $\mathcal{D}_2$ witnessing $q, p/q, R_0, \ldots, R_n \vdash_S r$. But as $\mathcal{D}_2$ contained no instances of p/q it witnesses $q, R_0, \ldots, R_n \vdash_S r$. The derivation $\mathcal{D}_2$ is the fourth proof we were looking for and we are done with $j = 0$. And so, we are done. □

We will use this result later in Lemma 5.5. This result doesn't hold when there are higher-level rules since discharged rules may be separated from the rule that discharges them.

Atomic rules can now be used to define proof-theoretic systems.

Definition 4.7. Let the set of all atomic rules of any level be denoted as $\mathbb{S}$. Call a set $\mathfrak{S} \subseteq \mathcal{P}(\mathbb{S})$ an *proof-theoretic system.*

We use these proof-theoretic systems in the next section as a base to define two consequence relations for proof-theoretic validity.

4.2 Prawitz's Definition of Proof-Theoretic Validity

Proof-Theoretic Validity was proposed by Prawitz (1971) as an explication of Gentzen's claim that:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'. (Gentzen 1935, p. 80)

There are well-known problems with taking this explanation literally. As Prawitz points out the introduction rules are not, in fact, explicit definitions, in the sense of definiendum-definiens pairs. Because of this Prawitz suggests, instead, that the introduction rules are used to inductively define a notion of 'validity'. In doing this, Prawitz is making the notion of consequence central to his interpretation of Gentzen. So, instead of showing that the elimination rules follow in some way from the introduction rules, it would instead be shown that they preserve 'validity'. In this definition, S is an atomic system and $\mathcal{J}$ is a set of transformations on derivations which preserve the conclusion and do not add open assumptions, though they may delete them.

Definition 4.8. (Prawitz 1973, p. 236; Schroeder-Heister 2006, p. 560; Schroeder-Heister 2018, Supplement 1) A derivation $\mathcal{D}$ being an *$(S, \mathcal{J})$-valid derivation* for an atomic system S and set of justifications $\mathcal{J}$ is defined inductively as follows:

(4.8.1.) If $\mathcal{D}$ is a closed derivation in S then it is $(S, \mathcal{J})$-valid.

(4.8.2.) If $\mathcal{D}$ is a closed derivation ending in an introduction rule then it is $(S, \mathcal{J})$-valid if the derivations of the premises of the introduction rule are $(S, \mathcal{J})$-valid.

(4.8.3.) If $\mathcal{D}$ is a closed derivation which does not end in an introduction rule then it is $(S, \mathcal{J})$-valid if it $\mathcal{J}$-reduces to an $(S, \mathcal{J})$-valid derivation which does end in an introduction rule.

(4.8.4.) If $\mathcal{D}$ is an open derivation of φ with open assumptions $\varphi_0, \ldots, \varphi_n$ then it is $(S, \mathcal{J})$-valid if for all atomic systems S' extending S, all justifications $\mathcal{J}'$ extending $\mathcal{J}$ and all closed $(S', \mathcal{J}')$-valid derivations $\mathcal{D}_0, \ldots, \mathcal{D}_n$ of $\varphi_0, \ldots, \varphi_n$, the following derivation is $(S', \mathcal{J}')$-valid:

$$\frac{\begin{matrix}\mathcal{D}_0 & \cdots & \mathcal{D}_n \\ \varphi_0 & \cdots & \varphi_n\end{matrix}}{\begin{matrix}\mathcal{D} \\ \varphi\end{matrix}}$$

Example 4.9. This definition can be illustrated by considering an example. Take the following proof of $p \to \neg\neg p$ or $p \to ((p \to \bot) \to \bot)$. Let $S = \varnothing$ and $\mathcal{J}$ be the standard reductions used in the proof of normalisation:

$$\dfrac{\dfrac{\dfrac{[p] \qquad [p \to \bot]}{\bot}}{(p \to \bot) \to \bot}}{p \to ((p \to \bot) \to \bot)}$$

This is a closed proof and it ends on an introduction rule so by condition 4.8.2 this proof is $(S, \mathcal{J})$-valid if its immediate sub-proof is:

$$\dfrac{\dfrac{p \qquad [p \to \bot]}{\bot}}{(p \to \bot) \to \bot}$$

This proof is open and so by condition 4.8.4 it is $(S, \mathcal{J})$-valid if given any closed proof of p in every extension S' and $\mathcal{J}'$ of S and $\mathcal{J}$ the proof generated by replacing the assumption p with its derivation is $(S', \mathcal{J}')$-valid. For the purposes of illustration let us consider the simplest case where we simply extend S with the axiom $/p$. By 4.8.1 the proof $/p$ is $(\{/p\}, \mathcal{J})$-valid. The new proof ends in an introduction rule and so by 4.8.2 will be $(\{/p\}, \mathcal{J})$-valid just if its immediate sub proof is:

$$\dfrac{\overline{p} \qquad p \to \bot}{\bot}$$

This is again an open proof so by condition 4.8.4 we need to consider all expansions of $\{/p\}$ and $\mathcal{J}$. This time there are two obvious expansions of $\{/p\}$. The first by $/\bot$ and the second by $\frac{p}{\bot}$. Let's use the second. We then get the following proof which we need to check is $(\{/p, \frac{p}{\bot}\}, \mathcal{J})$-valid:

$$\dfrac{\overline{p} \qquad \dfrac{\dfrac{[p]}{\bot}}{p \to \bot}}{\bot}$$

This proof does not end in an introduction rule, so we finally get to use condition 4.8.3. Careful inspection of the proof will show that we introduced $\to$ only to eliminate it directly afterwards. But we can use one of the reductions from the proof of normalisation to remove this. This results in the proof:

$$\dfrac{\overline{p}}{\bot}$$

Which is $(\{/p, \frac{p}{\bot}\}, \mathcal{J})$-valid by 4.8.1. To make this illustration into a rigorous argument, one would simply follow this line of reasoning backwards, starting with the small proofs and building up, and replacing our consideration of the simplest cases by the more general cases. But like with other inductive definitions, such as that of well-formed formula, sometimes they are best illustrated by breaking a familiar example down rather than building up from the base cases.

We can replace Prawitz's conditions on derivations with a consequence relation. To do this we need to define a consequence relation directly from derivations. We take this definition from Schroeder-Heister (2006, p. 561) who provides it in the course of explicating Prawitz's conjecture (Prawitz 1973, p. 246).

Definition 4.10. Given $S \in \mathfrak{S}$ let $\Gamma \Vdash_S^{\mathfrak{S}} \varphi$ hold if there is a derivation $\mathcal{D}$ of φ with open assumptions in Γ which is $(S, \mathcal{J}_{MAX})$-valid where $\mathcal{J}_{MAX}$ is the maximal set of justifications. Given a proof-theoretic system $\mathfrak{S}$ let the Prawitz semantics associated with it define the consequence relation $\Vdash_{\mathfrak{S}}$ where $\Gamma \Vdash_{\mathfrak{S}} \varphi$ if and only if for all $S \in \mathfrak{S}$ it follows $\Gamma \Vdash_S^{\mathfrak{S}} \varphi$.

Here we understand justifications in the sense of Schroeder-Heister (2006, p. 558) where a justification is any map from proofs to proofs, which preserve conclusions and does not add assumptions, that can be applied even if the proof is a sub-proof of a larger one. All the reductions used in normalization are justifications. As Schroeder-Heister points out, this differs from Prawitz's who places a constraint of 'consistency' on justifications. However, Schroeder-Heister argues that Prawitz's constraints are based on worries about normalization, not validity, and so they can be removed. With this we can show that the defined consequence relation meets the following conditions:

Lemma 4.11. Given a proof-theoretic system $\mathfrak{S}$, Prawitz's validity notion from Definition 4.10 satisfies the following:

$$\vdash_S p \Longleftrightarrow \Vdash_S^{\mathfrak{S}} p, \qquad \text{(Autonomy of Atoms)}$$

$$\Vdash_S^{\mathfrak{S}} \varphi \text{ and } \Vdash_S^{\mathfrak{S}} \psi \Longleftrightarrow \Vdash_S^{\mathfrak{S}} \varphi \wedge \psi, \qquad \text{(Conjunction Property)}$$

$$\Vdash_S^{\mathfrak{S}} \varphi \text{ or } \Vdash_S^{\mathfrak{S}} \psi \Longleftrightarrow \Vdash_S^{\mathfrak{S}} \varphi \vee \psi, \qquad \text{(Disjunction Property)}$$

$$[\forall S' \supseteq S(S' \in \mathfrak{S} \text{ and } \Vdash_{S'}^{\mathfrak{S}} \psi \Rightarrow \Vdash_{S'}^{\mathfrak{S}} \varphi)] \Longleftrightarrow \Vdash_S^{\mathfrak{S}} \psi \rightarrow \varphi. \qquad \text{(Weak Monotonicity)}$$

$$\exists \text{finite}\Delta \subseteq \Gamma[\forall S' \supseteq S(S' \in \mathfrak{S} \text{ and } \Vdash_{S'}^{\mathfrak{S}} \Delta \Rightarrow \Vdash_{S'}^{\mathfrak{S}} \varphi)] \Longleftrightarrow \Gamma \Vdash_S^{\mathfrak{S}} \varphi. \qquad \text{(Compact Monotonicity)}$$

Proof. (Autonomy of Atoms): By definition, if $\vdash_S p$ there is a derivation which is S-valid. Assume there is a closed S-valid derivation $\mathcal{D}$ of an atomic proposition p. Now $\mathcal{D}$ is either in S, ends in an introduction rule, or reduces to a proof $\mathcal{D}'$ meeting one of the previous two conditions. So, we can assume we have a derivation either ending in an introduction rule or a derivation in S. But it cannot end with an introduction rule as it is a proof of an atomic formula. This leaves only that it is in S and so $\Vdash_S^{\mathfrak{S}} p \Leftrightarrow \vdash_S p$.

The proof of the conjunction and disjunction properties are relatively simple and so we omit them here.

(Compact Monotonicity): It needs to be shown that $\Gamma \Vdash_S^{\mathfrak{S}} \varphi$ if and only if $\exists finite \Delta \subseteq \Gamma \forall S' \supseteq S[\Vdash_{S'}^{\mathfrak{S}} \Delta \Rightarrow \Vdash_{S'}^{\mathfrak{S}} \varphi]$. Assume $\Gamma \Vdash_S^{\mathfrak{S}} \varphi$ then we know that there is some finite subset $\Delta = \{\psi_0, \dots, \psi_n\}$ and a derivation

$$\begin{array}{c} \psi_0, \dots, \psi_n \\ \mathcal{D} \\ \varphi \end{array}$$

and given any S' extending S and any closed S'-valid derivations $\begin{array}{c} \mathcal{D}_i \\ \psi_i \end{array}$ then the closed proof

$$\begin{array}{ccc} \mathcal{D}_0 & & \mathcal{D}_n \\ \psi_0 & \dots & \psi_n \\ & \mathcal{D} & \\ & \varphi & \end{array}$$

is S' valid. From which monotonicity clearly follows.

So now assume that there is finite $\Delta \subseteq \Gamma$ and for every S' extending S if there are closed S'-valid derivations of Δ then there is a closed S'-valid derivation of φ. Assume for a contradiction that $\Gamma \Vdash_S^{\mathfrak{S}} \varphi$ is false. What follows from this is that there is no derivation in S with open assumptions in Γ that satisfy Definition 4.8.4, including

$$\frac{\Delta}{\varphi}.$$

Since $\frac{\Delta}{\varphi}$ does not satisfy Definition 4.8.4, there is a S' extending S such that there are S'-valid derivations $\mathcal{D}_i$ of every formula $\psi_i \in \Delta$ but the composition of those derivations with the original derivation of φ:

$$\frac{\begin{array}{ccc}\mathcal{D}_0 & & \mathcal{D}_n \\ \psi_0 & \dots & \psi_n\end{array}}{\varphi}$$

is not S'-valid. However, we have all justifications in our system, and so we have the justification that simply takes this derivation to *any* S'-valid derivation of φ. So, this comes down to the idea that there isn't a derivation of φ in S'. But as there are S'-valid derivations of Δ we know there is a derivation of φ.

(Weak Monotonicity): By Compact Monotonicity it is sufficient to show that $\psi \Vdash_S^{\mathfrak{S}} \varphi$ if and only if $\Vdash_S^{\mathfrak{S}} \psi \to \varphi$.

Assume $\psi \Vdash_S^{\mathfrak{S}} \varphi$ then there is an open S-valid derivation of ψ from premise φ. It follows that by condition 2 of Definition 4.8 the following is an S-valid derivation of $\varphi \to \psi$:

$$\frac{\begin{array}{c}[\varphi] \\ \mathcal{D} \\ \psi\end{array}}{\varphi \to \psi}$$

so $\Vdash_S^{\mathfrak{S}} \psi \to \varphi$.

Assume $\Vdash_S^{\mathfrak{S}} \psi \to \varphi$ then there is a S-valid derivation $\mathcal{D}$ of $\varphi \to \psi$. As $\varphi \to \psi$ is not an atomic formula it follows that either condition 3 holds of $\mathcal{D}$ and there is a justification taking it to a derivation ending in an introduction rule, or condition 2 holds and $\mathcal{D}$ ends in an introduction rule. So, we assume condition 2 holds. Then $\mathcal{D}$ is of the form:

$$\frac{\begin{array}{c}[\varphi] \\ \mathcal{D} \\ \psi\end{array}}{\varphi \to \psi} \quad \text{From this it follows that:} \quad \begin{array}{c}\varphi \\ \mathcal{D} \\ \psi\end{array}$$

is an S-valid derivation of ψ from φ so $\psi \Vdash_S^{\mathfrak{S}} \varphi$. □

It can also be shown that any two consequence relations satisfying the conditions in Lemma 4.11 for a Prawitz semantics will be identical.

Lemma 4.12. Given a proof-theoretic system $\mathfrak{S}$ and two binary relations satisfying the conditions of Lemma 4.11, $(\mathfrak{S}, \Vdash_{\mathfrak{S}})$ and $(\mathfrak{S}, \Vdash'_{\mathfrak{S}})$, it follows that $\Vdash_S^{\mathfrak{S}} = \Vdash_S'^{\mathfrak{S}}$ for all $S \in \mathfrak{S}$.

Though we do not include the proof here, it is a simple induction on formula complexity.

4.3 PCS Semantics

While the above formulation is essentially that found in Prawitz's work, most of the results on the logic of PTV use a different formulation. We call it PCS semantics because it appears in Piecha, Campos Sanz, and Schroeder-Heister (2015). It is PCS semantics that we will use for the rest of the paper. The two definitions differ only on the condition for $\Gamma \vDash_S^{\mathfrak{S}} \varphi$. While it appears not to have been addressed in the literature, we will show that this notion is not equivalent to Prawitz's. It is hoped that in future work we will be able to show that the definitions coincide on the relevant proof-theoretic systems.

Definition 4.13. A pair $(\mathfrak{S}, \vDash_{\mathfrak{S}})$ is a PCS semantics if $\mathfrak{S}$ is a proof-theoretic system (Definition 4.7), and if for every S in $\mathfrak{S}$ the relation $\vDash_S^{\mathfrak{S}}$ satisfies the following:

$$\vdash_S p \Longleftrightarrow \vDash_S^{\mathfrak{S}} p, \qquad \text{(Autonomy of Atoms)}$$

$$\vDash_S^{\mathfrak{S}} \varphi \text{ and } \vDash_S^{\mathfrak{S}} \psi \Longleftrightarrow \vDash_S^{\mathfrak{S}} \varphi \wedge \psi, \qquad \text{(Conjunction Property)}$$

$$\vDash_S^{\mathfrak{S}} \varphi \text{ or } \vDash_S^{\mathfrak{S}} \psi \Longleftrightarrow \vDash_S^{\mathfrak{S}} \varphi \vee \psi, \qquad \text{(Disjunction Property)}$$

$$[\forall S' \supseteq S(S' \in \mathfrak{S} \text{ and } \vDash_{S'}^{\mathfrak{S}} \psi \Rightarrow \vDash_{S'}^{\mathfrak{S}} \varphi)] \Longleftrightarrow \vDash_S^{\mathfrak{S}} \psi \to \varphi. \qquad \text{(Weak Monotonicity)}$$

$$[\forall S' \supseteq S(S' \in \mathfrak{S} \text{ and } \vDash_{S'}^{\mathfrak{S}} \Gamma \Rightarrow \vDash_{S'}^{\mathfrak{S}} \varphi)] \Longleftrightarrow \Gamma \vDash_S^{\mathfrak{S}} \varphi. \qquad \text{(Monotonicity)}$$

and further $\vDash_{\mathfrak{S}}$ is defined from $\vDash_S^{\mathfrak{S}}$ as follows:

$$\Gamma \vDash_{\mathfrak{S}} \varphi \Longleftrightarrow \forall S \in \mathfrak{S}, \Gamma \vDash_S^{\mathfrak{S}} \varphi. \tag{17}$$

Each proof-theoretic system has a unique PCS semantics.

Lemma 4.14. Given a proof-theoretic system $\mathfrak{S}$ there is a consequence relation $\vDash_{\mathfrak{S}}$ such that $(\mathfrak{S}, \vDash_{\mathfrak{S}})$ is a PCS semantics and given two PCS semantics $(\mathfrak{S}, \vDash_{\mathfrak{S}})$ and $(\mathfrak{S}, \vDash'_{\mathfrak{S}})$ it follows that $\vDash_S^{\mathfrak{S}} = \vDash_S'^{\mathfrak{S}}$ for $S \in \mathfrak{S}$.

All PCS semantics are extensions of minimal logic. If the proof-theoretic system includes $\bot/p$ for all atomic formulas p then it will satisfy intuitionistic logic. This follows from the following correspondence with Kripke models:

Lemma 4.15. Given a PCS semantics $(\mathfrak{S}, \vDash_{\mathfrak{S}})$ such that every $S \in \mathfrak{S}$ contains $\bot/p$ for all atomic formulas p, the Kripke semantics $\langle \mathfrak{S} - \{S \mid \vdash_S \bot\}, \subseteq, V\rangle$ where $V(p) = \{S \mid \vdash_S p\}$ defines the same satisfaction relation.

Proof. Proof by induction on complexity of formulas. Let $S \in \mathfrak{S} - \{S \mid \vdash_S \bot\}$ then the base case, conjunction and disjunction case are trivial. Assume $\langle \mathfrak{S} - \{S \mid \vdash_S \bot\}, \subseteq, V\rangle, S \Vdash \varphi \to \psi$. It will follow that $\vDash_S^{\mathfrak{S}} \varphi \to \psi$ as all $S' \in \mathfrak{S} - \{S \mid \vdash_S \bot\}$ are also in $\mathfrak{S}$ and for all $S' \in \mathfrak{S}$ and not in $\mathfrak{S} - \{S \mid \vdash_S \bot\}$ it follows that $\vDash_{S'}^{\mathfrak{S}} \chi$ for all χ as $\vdash_{S'} \bot$. Assume $\vDash_S^{\mathfrak{S}} \varphi \to \psi$ it follows that for all $S' \in \mathfrak{S}$ if $S' \supseteq S$ and $\vDash_{S'}^{\mathfrak{S}} \varphi$ then $\vDash_{S'}^{\mathfrak{S}} \psi$. Take $S^* \in \mathfrak{S} - \{S \mid \vdash_S \bot\}$ such that $S^* \supseteq S$ and $\langle \mathfrak{S} - \{S \mid \vdash_S \bot\}, \subseteq, V\rangle, S^* \Vdash \varphi$ then by the induction hypothesis $\langle \mathfrak{S} - \{S \mid \vdash_S \bot\}, \subseteq, V\rangle, S^* \Vdash \psi$. □

It can be shown that PCS and Prawitz notion from Section 4.2 are distinct because Monotonicity isn't equivalent to Compact Monotonicity.

Lemma 4.16. Let $\mathfrak{S} = \{\{p_0, \ldots, p_n\} \mid n \in \mathbb{N}\} \cup \{\{q, p_0, p_1, \ldots\}\}$ then $\{p_0, p_1, \ldots\} \vDash_{\mathfrak{S}} q$ but $\{p_0, p_1, \ldots\} \nVdash_{\mathfrak{S}} q$.

Proof. Note that $\{p_0, p_1, \dots\} \vDash_{\mathfrak{S}} q$ if for all $S \in \mathfrak{S}$ when $\vDash^{\mathfrak{S}}_S \{p_0, p_1, \dots\}$ then $\vDash^{\mathfrak{S}}_S q$. As the only S such that $\vDash^{\mathfrak{S}}_S \{p_0, p_1, \dots\}$ is $\{q, p_0, p_1, \dots\}$ this follows. However, $\{p_0, p_1, \dots\} \Vdash_{\mathfrak{S}} q$ if there is a derivation $\mathcal{D}$ of q with (finite) open assumptions in $\{p_0, p_1, \dots\}$ such that any S which has a derivation of those open assumptions has a derivation of q. Assume this is so for a contradiction. Then there is some n such that $\{p_0, \dots, p_n\}$ contains all the open assumptions in $\mathcal{D}$. But then $S = \{p_0, \dots, p_n\}$ has a derivation of $p_0, \dots, p_n$ and yet it contains (by there being no rules ending in q) no derivation of q. It follows that $\{p_0, p_1, \dots\} \nVdash_{\mathfrak{S}} q$. □

The reason for this is that while $\vDash$ has the monotonicity condition $[\forall S' \supseteq S(\vDash^{\mathfrak{S}}_{S'} \Gamma \Rightarrow \vDash^{\mathfrak{S}}_{S'} \varphi)] \iff \Gamma \vDash^{\mathfrak{S}}_S \varphi$, for $\Vdash$ there is instead compact monotonicity condition $\exists \text{finite}\Delta \subseteq \Gamma[\forall S' \supseteq S(\vDash^{\mathfrak{S}}_{S'} \Delta \Rightarrow \vDash^{\mathfrak{S}}_{S'} \varphi)] \iff \Gamma \vDash^{\mathfrak{S}}_S \varphi$. A straightforward result of this proof is that $\Vdash$ does not always satisfy monotonicity as it still holds in the example above that if $\Vdash^{\mathfrak{S}}_S \{p_0, p_1, \dots\}$, then $\Vdash^{\mathfrak{S}}_S q$. However, when monotonicity does hold Lemma 4.11 straightforwardly gives us the following result:

Lemma 4.17. Given a proof-theoretic system $\mathfrak{S}$ for which $\Vdash$ is monotone, then for all $S \in \mathfrak{S}$ and φ and Γ we have

$$\Gamma \Vdash^{\mathfrak{S}}_S \varphi \Leftrightarrow \Gamma \vDash^{\mathfrak{S}}_S \varphi$$

Proof. By Lemma 4.11 and the assumption of monotonicity we know that it meets the criteria to be a PCS semantics and then by Lemma 4.14 we know that any two PCS semantics with the same proof-theoretic system are identical. □

On the other side of the equation what is lacking is compactness. That is $\Gamma \vDash^{\mathfrak{S}}_S \varphi \Leftrightarrow \exists \text{finite}\Delta \subseteq \Gamma\ \Delta \vDash^{\mathfrak{S}}_S \varphi$. It can be shown that whenever the PCS semantics is compact it agrees with Prawitz's notion.

Lemma 4.18. Let $\vdash_L$ be compact and monotonic then it has compact monotonicity.

Proof. $\Gamma \vdash_L \varphi \Leftrightarrow_{\text{Compact.}} \exists \text{finite}\Delta \subseteq \Gamma\ \Delta \vdash_L \varphi \Leftrightarrow_{\text{Monoton.}} \exists \text{finite}\Delta \subseteq \Gamma[\forall S' \supseteq S(\Vdash^{\mathfrak{S}}_{S'} \Delta \Rightarrow \Vdash^{\mathfrak{S}}_{S'} \varphi)]$. □

This allows us to show:

Lemma 4.19. Given a proof-theoretic system $\mathfrak{S}$ for which $\vDash$ is compact, then for all $S \in \mathfrak{S}$ and φ and Γ we have

$$\Gamma \vDash^{\mathfrak{S}}_S \varphi \Leftrightarrow \Gamma \Vdash^{\mathfrak{S}}_S \varphi$$

Proof. Given that $\vDash^{\mathfrak{S}}_S$ is compact, it follows that it has compact monotonicity by Lemma 4.18. Then by Lemma 4.12 it follows that they must be identical. □

We do not currently know which proof-theoretic systems are compact or monotone. And so, we do not know on which proof-theoretic systems PCS semantics and Prawitz's validity notion align. Characterising the set of proof-theoretic systems which are compact will be the subject of future work. For our purposes, we continue to work with PCS semantics given the usefulness of monotonicity. When we move to the weak logics defined by these systems we will have compactness built-in.

5 Results about PCS semantics

5.1 Relations between proof-theoretic systems

When considering the logic of PTV, the proof-theoretic system which is chosen can have a considerable impact on the logic that results. We consider $\mathfrak{S}^M_\infty = \mathcal{P}(\mathbb{S})$ to be the largest possible proof-theoretic system. Here M stands for minimal as there is no special treatment of $\bot$ in this system. The system $\mathfrak{S}_\infty = \{S \subseteq \mathbb{S} \mid (\bot/p) \in S \text{ for all atomic } p\}$ restricts $\mathfrak{S}^M_\infty$ to ensure that $\bot$ behaves as it does in intuitionistic logic. And when we refer to the logic of PTV, it is the logic of this system that we mean. All other proof-theoretic systems are defined as subsets of these proof-theoretic systems. The following result is worth noting:

Lemma 5.1. Let φ be $\to$ free and $\mathfrak{S}' \subseteq \mathfrak{S}$, then for all $S \in \mathfrak{S}'$

$$\vDash^{\mathfrak{S}}_S \varphi \Leftrightarrow \vDash^{\mathfrak{S}'}_S \varphi$$

Proof. For the atomic case note that: $\vDash^{\mathfrak{S}}_S p \Leftrightarrow \vdash_S p \Leftrightarrow \vDash^{\mathfrak{S}'}_S p$. The induction cases follow immediately from the induction hypothesis. □

Corollary 5.2. Let φ be $\to$ free and $\mathfrak{S}' \subseteq \mathfrak{S}$, then

$$\vDash_{\mathfrak{S}} \varphi \Rightarrow \vDash_{\mathfrak{S}'} \varphi$$

Given a proof-theoretic system $\mathfrak{S}$ and an atomic system $K \in \mathfrak{S}$, two natural ways to get new proof-theoretic systems are as follows:

1. $\{S \in \mathfrak{S} \mid S \subseteq K\}$

   This defines an ideal on $\mathfrak{S}$. Examples are $\mathfrak{S}^M_n$ and $\mathfrak{S}_n$, where these are $\mathfrak{S}^M_n = \{S \in \mathfrak{S}^M_\infty \mid S \subseteq \{R \mid R \text{ is level } n\}\}$ and $\mathfrak{S}_n = \{S \in \mathfrak{S}_\infty \mid S \subseteq \{R \mid R \text{ is level } n\}\}$. That is $\mathfrak{S}^M_\infty$ and $\mathfrak{S}_\infty$ restricted to rules of at most level n.

2. $\{S \in \mathfrak{S} \mid K \subseteq S\}$

   This defines a filter on $\mathfrak{S}$. An example is $\mathfrak{S}_\infty$ which is $\{S \in \mathfrak{S}^M_\infty \mid \{(\bot/p) \mid p \text{ atomic}\} \subseteq S\}$. In other words, $\mathfrak{S}_\infty$ is every $S \in \mathfrak{S}^M_\infty$ such that $(\bot/p) \in S$ for every atomic p.

A proof-theoretic system $\mathfrak{S}' \subseteq \mathfrak{S}$ is closed under supersets from $\mathfrak{S}$ if whenever $S \in \mathfrak{S}'$, $S' \in \mathfrak{S}$ and $S \subseteq S'$ then $S' \in \mathfrak{S}'$. Filters are closed under supersets from $\mathfrak{S}$. If a proof-theoretic system is a subset of another and is closed under supersets from it, then it proves anything the larger proof-theoretic system proves:

Lemma 5.3. If $\mathfrak{S}' \subseteq \mathfrak{S}$ is closed under supersets from $\mathfrak{S}$, then for all $S \in \mathfrak{S}'$

$$\vDash^{\mathfrak{S}}_S \varphi \Leftrightarrow \vDash^{\mathfrak{S}'}_S \varphi$$

Proof. Induction on φ. All cases except for $\to$ are covered by the inductive steps in the proof of Lemma 5.1. For the induction hypothesis, assume we have for all $S \in \mathfrak{S}'$ that $\vDash^{\mathfrak{S}}_S \varphi \Leftrightarrow \vDash^{\mathfrak{S}'}_S \varphi$ and $\vDash^{\mathfrak{S}}_S \psi \Leftrightarrow \vDash^{\mathfrak{S}'}_S \psi$. Assume $\vDash^{\mathfrak{S}}_{S'} \varphi \to \psi$ and $S' \in \mathfrak{S}'$, then by the weak monotonicity and monotonicity of Definition 4.13 it follows that for all $S \in \mathfrak{S}$ extending S' if $\vDash^{\mathfrak{S}}_S \varphi$ then $\vDash^{\mathfrak{S}}_S \psi$. Take $S'' \in \mathfrak{S}'$ extending S' such that $\vDash^{\mathfrak{S}'}_{S''} \varphi$. Clearly $S'' \in \mathfrak{S}$. Now by the induction hypothesis $\vDash^{\mathfrak{S}}_{S''} \varphi$ from which it follows that $\vDash^{\mathfrak{S}}_{S''} \psi$ and so again by the induction hypothesis $\vDash^{\mathfrak{S}'}_{S''} \psi$.

Assume $\vDash_S^{\mathfrak{S}'} \varphi \to \psi$, then for all $S' \in \mathfrak{S}'$ extending S such that $\vDash_{S'}^{\mathfrak{S}'} \varphi$ it follows that $\vDash_{S'}^{\mathfrak{S}'} \psi$. Take $S' \in \mathfrak{S}$ extending S such that $\vDash_{S'}^{\mathfrak{S}} \varphi$. As S' extends $S \in \mathfrak{S}'$ it follows by assumption of closure under supersets that $S' \in \mathfrak{S}'$. Now by the induction hypothesis $\vDash_{S'}^{\mathfrak{S}'} \varphi$ from which it follows that $\vDash_{S'}^{\mathfrak{S}'} \psi$ and so again by the induction hypothesis $\vDash_{S'}^{\mathfrak{S}} \psi$. $\square$

Lemma 5.4. If $\mathfrak{S}' \subseteq \mathfrak{S}$ is closed under supersets from $\mathfrak{S}$, then

$$\vDash_{\mathfrak{S}} \varphi \Rightarrow \vDash_{\mathfrak{S}'} \varphi$$

Proof. Assume $\vDash_{\mathfrak{S}} \varphi$ then for all $S \in \mathfrak{S}$, $\vDash_S^{\mathfrak{S}} \varphi$. As all $S' \in \mathfrak{S}'$ are also in $\mathfrak{S}$ it follows by Lemma 5.3 that $\vDash_{S'}^{\mathfrak{S}'} \varphi$ and so $\vDash_{\mathfrak{S}'} \varphi$. $\square$

Straight away we get from this that as $\mathfrak{S}_\infty$ is a filter on $\mathfrak{S}_\infty^M$ it follows that $\vDash_{\mathfrak{S}_\infty^M} \varphi$ implies $\vDash_{\mathfrak{S}_\infty} \varphi$ and so the 'intuitionistic' system prove everything the 'minimal' one does. The hypothesis of closure under supersets is crucial for Lemma 5.4. This can be shown by considering $\mathfrak{S}_1$, the proof-theoretic system with only level 1 rules, and $\mathfrak{S}_\infty$. Note that $\mathfrak{S}_1 \subseteq \mathfrak{S}_\infty$ but $\mathfrak{S}_1$ is not closed under supersets in $\mathfrak{S}_\infty$. Every instance of the generalised Kreisel-Putnam axiom is provable in $\mathfrak{S}_\infty$ (see Lemma 5.12) but we can construct instances of the generalised Kreisel-Putnam axiom that are not provable in $\mathfrak{S}_1$, as we show in the proof below. So, we see that the smaller proof-theoretic system, $\mathfrak{S}_1$, does not prove everything the larger one, $\mathfrak{S}_\infty$, does.

Lemma 5.5. For distinct r, p, q not equal to falsum: $\vDash_{\mathfrak{S}_1} ((p \to q) \to r) \to (r \vee (p \wedge (q \to r)))$ but $\nvDash_{\mathfrak{S}_1} ((p \to q) \to r) \to r$ and $\nvDash_{\mathfrak{S}_1} ((p \to q) \to r) \to (p \wedge (q \to r))$.

Proof. First we show that $\vDash_{\mathfrak{S}_1} ((p \to q) \to r) \to (r \vee (p \wedge (q \to r)))$. This follows if for all $S \in \mathfrak{S}_1$ if $\vDash_S^{\mathfrak{S}_1} ((p \to q) \to r)$ then $\vDash_S^{\mathfrak{S}_1} r \vee (p \wedge (q \to r))$. Let $S \in \mathfrak{S}_1$ be such that $\vDash_S^{\mathfrak{S}_1} ((p \to q) \to r)$. Now either $\vdash_S r$ or not. If $\vdash_S^{\mathfrak{S}_1} r$ then clearly $\vDash_S^{\mathfrak{S}_1} r \vee (p \wedge (q \to r))$. So, assume not. We know that $\vdash_{S \cup \{p/q\}}^{\mathfrak{S}_1} r$ because $\vDash_{S \cup \{p/q\}}^{\mathfrak{S}_1} p \to q$. So as we have at most level 1 rules there must be proofs, by Lemma 4.6, such that $\vdash_S p$ and $q \vdash_S r$, so $\vDash_S^{\mathfrak{S}_1} r \vee (p \wedge (q \to r))$.

But $\nvDash_{\mathfrak{S}_1} ((p \to q) \to r) \to r$ as $S = \{/p, q/r\} \cup \{\bot/a \mid a \text{ atomic}\}$ demonstrates. Note that $\vDash_S^{\mathfrak{S}_1} ((p \to q) \to r)$. Because assuming S' extending S proves $\vDash_{S'}^{\mathfrak{S}_1} p \to q$ then as $/p \in S'$ it follows that $\vDash_{S'}^{\mathfrak{S}_1} q$ and as $q/r \in S'$ then $\vDash_{S'}^{\mathfrak{S}_1} r$. We need to show that $\nvDash_S^{\mathfrak{S}_1} r$ which is the case only if $\nvdash_S r$. Assume $\vdash_S r$ for a contradiction, then there is a derivation ending in r but this can be the case only by application of q/r. So, we would have a derivation of q but no rule ends in q and so no such demonstration is available. And $\nvDash_{\mathfrak{S}_1} ((p \to q) \to r) \to (p \wedge (q \to r))$ as $S = \{/r\} \cup \{\bot/a \mid a \text{ atomic}\}$ demonstrates. $\square$

We can give an even simpler example of how the adding of systems can radically alter what a proof-theoretic system proves. Consider the proof-theoretic systems $\{\varnothing\}$, $\{\{p/q\}\}$, $\{\{/p\}\}$, $\{\{/p, p/q\}\}$, $\{\varnothing,\ \{p/q\},\ \{/p\},\ \{/p, p/q\}\}$. Clearly the first four are subsets of the fifth one. And yet no clear relationship holds between what is true on the smaller proof-theoretic systems and the larger system.

	$\vDash_{\{\varnothing\}}$	$\vDash_{\{\{p/q\}\}}$	$\vDash_{\{\{/p\}\}}$	$\vDash_{\{\{/p,p/q\}\}}$	$\vDash_{\{\varnothing,\{p/q\},\{/p\},\{p/q,/p\}\}}$
p	no	no	yes	yes	no
$p \to q$	yes	yes	no	yes	no
$(p \to q) \to p$	no	no	yes	yes	no

This highlights both an interesting feature of proof-theoretic systems and why they can be so hard to work with.

5.2 $\mathfrak{S}_\infty$ Satisfies the Kreisel-Putnam axiom

Piecha, Campos Sanz, and Schroeder-Heister (2015) show that we can replace any disjunction-free formula on the left-hand side of $\vDash_S^{\mathfrak{S}_\infty}$ with a rule in $\mathbb{S}$ and vice-versa. This result will help us to define proof-theoretic systems with particular sets of formulas in their logic. So, we will spell it out in some detail. The first thing that needs to be done is to define rule formulas, which are formulas that have a direct correspondence to atomic rules.

Definition 5.6. We define *rule formulas* and their associated *levels* inductively as follows:

- an atomic formula p is a rule-formula of level 0
- where $\varphi_1, \dots, \varphi_n$ are rule formulas of level $i_1, \dots, i_n$ and p is an atomic formula, then $(\varphi_1 \wedge \dots \wedge \varphi_n) \to p$ is a rule-formula of level $\max(i_1, \dots, i_n) + 1$.

It turns out that all disjunction free formulas are equivalent to atomic rules. This is because each disjunction free formula is equivalent to a rule formula.

Lemma 5.7. (Piecha, Campos Sanz, and Schroeder-Heister 2015) In IPC, every disjunction-free formula is equivalent to a conjunction of rule-formulas.

Proof. We must show that for every disjunction free formula ψ there are rule-formulas $\varphi_1, \dots, \varphi_n$ such that $IPC \vdash \psi \equiv \varphi_1 \wedge \dots \wedge \varphi_n$. Base case: Assume p is an atomic formula then p is a rule-formula and $IPC \vdash p \equiv p$. Induction case: There are no inductive steps associated with disjunction since we are working under the assumption of disjunction freeness and the conjunction case is trivial. The only difficult case is the inductive step associated to $\varphi \to \psi$. We know $IPC \vdash (\varphi \to \psi) \equiv (\varphi_1 \wedge \dots \wedge \varphi_n \to \psi_1 \wedge \dots \wedge \psi_m)$ and $IPC \vdash (\chi \to \theta_1 \wedge \theta_2) \equiv ((\chi \to \theta_1) \wedge (\chi \to \theta_2))$. So, it is sufficient to show that each $\varphi_1 \wedge \dots \wedge \varphi_n \to \psi_i$ is equivalent to a rule-formula. Note that as $\psi_1 \wedge \dots \wedge \psi_m$ are rule-formulas we know that either they are atomic formulas or of the form $(\chi_1 \wedge \dots \wedge \chi_j) \to q$. If the former we are done so assume $\psi_i = (\chi_1 \wedge \dots \wedge \chi_j) \to q$. But as $IPC \vdash (\theta_1 \to (\theta_2 \to \theta_3)) \equiv (\theta_1 \wedge \theta_2 \to \theta_3)$, it follows that $IPC \vdash (\varphi_1 \wedge \dots \wedge \varphi_n) \to ((\chi_1 \wedge \dots \wedge \chi_j) \to q) \equiv ((\varphi_1 \wedge \dots \wedge \varphi_n \wedge \chi_1 \wedge \dots \wedge \chi_j) \to q)$ which is a rule-formula. □

We will now spell out the translation between atomic rules and rule formulas. In what follows if R is of the form p then R/r is p/r and if R is of the form $[R']q/p$ with $[R']$ possibly empty then R/r is $[R'/q]p/r$. With this we associate rules with rule formulas one-to-one as follows:

1. $/p^* = p$
2. $(p_0, \dots, p_n/q)^* = p_0 \wedge \dots \wedge p_n \to q$
3. If $R_{0,0}, \dots, R_{0,m_0}, \dots, R_{n,0}, \dots, R_{n,m_n}$ are associated to $\varphi_{0,0}, \dots, \varphi_{0,m_0}, \dots, \varphi_{n,0}, \dots, \varphi_{n,n_m}$ then

   $(([R_{0,0}, \dots, R_{0,m_0}]q_0), \dots, ([R_{n,0}, \dots, R_{n,m_n}]q_n)/r)^* = (((\varphi_{0,0} \wedge \dots \wedge \varphi_{0,m_0}) \to q_0) \wedge \dots \wedge ((\varphi_{n,0} \wedge \dots \wedge \varphi_{n,m_n}) \to q_n) \to r)$

We associate rule formulas with rules one-to-one as follows:

4. $p^+ = /p$
5. If $\varphi_0, \dots, \varphi_n$ are associated to $R_0, \dots, R_n$ then $((\varphi_0 \wedge \dots \wedge \varphi_n) \to r)^+ = ((R_0, \dots, R_n)/r)$.

It can be checked that $\varphi^{+*} = \varphi$ and $R^{*+} = R$. Let $S^* = \{R^* \mid R \in S\}$ and $\Gamma^+ = \{\varphi^+ \mid \varphi \in \Gamma\}$. Given this we have the following result from Piecha, Campos Sanz, and Schroeder-Heister (2015):

Lemma 5.8 (Piecha, Campos Sanz, and Schroeder-Heister 2015, Lemma 2 (C4))**.** Let Δ be a set of disjunction-free formulas.

$$\Gamma, \Delta \vDash_C^{\mathfrak{S}_\infty} \varphi \Leftrightarrow \Gamma \vDash_{C\cup\Delta^+}^{\mathfrak{S}_\infty} \varphi \text{ and } \Gamma, S^* \vDash_C^{\mathfrak{S}_\infty} \varphi \Leftrightarrow \Gamma \vDash_{C\cup S}^{\mathfrak{S}_\infty} \varphi$$

We have been talking about logics with the disjunction property, but it turns out there is a generalisation of this property, which for PCS semantics implies the Kreisel-Putnam axiom.

Definition 5.9. A consequence relation $\vDash$ has the generalised disjunction property if for all sets Γ of disjunction free formulas:

$$\Gamma \vDash \varphi \vee \psi \Leftrightarrow [\Gamma \vDash \varphi \text{ or } \Gamma \vDash \psi].$$

We can now show that $\mathfrak{S}_\infty$ satisfies the general disjunction property:

Lemma 5.10. (Piecha and Schroeder-Heister 2019) For all $S \in \mathfrak{S}_\infty$, the consequence relation $\vDash_S^{\mathfrak{S}_\infty}$ has the generalised disjunction property.

Proof. Let Γ be disjunction free and $\Gamma \vDash_S^{\mathfrak{S}_\infty} \varphi \vee \psi$. By Lemmas 4.15 we know that $\vDash_S^{\mathfrak{S}}$ satisfies IPC, so by Lemma 5.7 we can assume that Γ is a set of rule-formulas. By Lemma 5.8, $\vDash_{S\cup\Gamma^+}^{\mathfrak{S}_\infty} \varphi \vee \psi$. So, by the disjunction property from Definition 4.13 $\vDash_{S\cup\Gamma^+}^{\mathfrak{S}_\infty} \varphi$ or $\vDash_{S\cup\Gamma^+}^{\mathfrak{S}_\infty} \psi$. Then $\Gamma \vDash_S^{\mathfrak{S}_\infty} \varphi$ or $\Gamma \vDash_S^{\mathfrak{S}_\infty} \psi$ follows by Lemma 5.8. ☐

It follows that:

Corollary 5.11. (Piecha and Schroeder-Heister 2019) $\mathfrak{S}_\infty$ has the generalised disjunction property.

(Sketch). Assume $\Gamma \vDash_{\mathfrak{S}_\infty} \varphi \vee \psi$. It follows that for all $S \in \mathfrak{S}_\infty$ that $\Gamma \vDash_S^{\mathfrak{S}_\infty} \varphi \vee \psi$. In particular, let $S_\bot = \{\bot/p \mid p \text{ atomic}\}$ then $\Gamma \vDash_{S_\bot}^{\mathfrak{S}_\infty} \varphi \vee \psi$. Assume without loss of generality that $\Gamma \vDash_{S_\bot}^{\mathfrak{S}_\infty} \varphi$. It is a simple induction to show for all $S \in \mathfrak{S}_\infty$ that $\Gamma \vDash_S^{\mathfrak{S}_\infty} \varphi$. ☐

This means that $\mathfrak{S}_\infty$ has the disjunction property (trivially). Piecha and Schroeder-Heister (2019, Lemma 2.1 p. 4) have shown, any PCS semantics with the generalised disjunction property satisfies the generalised Kreisel-Putnam axiom.

Lemma 5.12. If a PCS semantics has the generalised disjunction property then it satisfies the generalised Kreisel-Putnam axiom.

Proof. Let $S, S' \in \mathfrak{S}$ and $S \supseteq S'$. Let $\vDash_S^{\mathfrak{S}} \varphi \to \psi_1 \vee \psi_2$ such that φ is disjunction free. It follows that $\varphi \vDash_S^{\mathfrak{S}} \psi_1 \vee \psi_2$ by the weak deduction theorem. By the generalised disjunction property $\varphi \vDash_S^{\mathfrak{S}} \psi_i$ for some $i \in \{1, 2\}$. So $\vDash_S^{\mathfrak{S}} \varphi \to \psi_i$. From which it follows that $\vDash_S^{\mathfrak{S}} (\varphi \to \psi_1) \vee (\varphi \to \psi_2)$. Which by monotonicity implies $\vDash_{S'}^{\mathfrak{S}} [\varphi \to \psi_1 \vee \psi_2] \to [(\varphi \to \psi_1) \vee (\varphi \to \psi_2)]$. ☐

We now have everything we need in place to define quasi-PTV and show that $\mathfrak{S}_\infty$ aligns with general inquisitive semantics.

6 Proof and Consequences of Main Theorem

Our goal is to show the equivalence of the two systems of inquisitive semantics we considered in Section 3 with proof-theoretic systems of proof-theoretic validity. To do this we need to ensure that the proof-theoretic systems meet the conditions set out in Corollary 2.11 and 2.12. We have already seen that $\mathfrak{S}_\infty$ satisfies the generalised disjunction property and the generalised Kreisel-Putnam axiom. Let $L_{\mathfrak{S}} = \{\varphi \mid \vDash_{\mathfrak{S}} \varphi\}$. Then the equality of $L_{\mathfrak{S}_\infty}$ and general inquisitive logic will follow if $L_{\mathfrak{S}_\infty}$'s disjunction free fragment is the same as intuitionistic logic's. Further, if we could define a filter on $\mathfrak{S}_\infty$ that satisfies double negation elimination for atomic formulas we would show that the logic of the defined proof-theoretic system was inquisitive logic.

Our first result follows from the following lemma:

Lemma 6.1 (Piecha, Campos Sanz, and Schroeder-Heister 2015, p. 331 Lem. 4)**.** For all disjunction free φ, $\vDash_{\mathfrak{S}_\infty} \varphi \Leftrightarrow \vdash_{IPC} \varphi$.

Proof. Assume Γ, φ are disjunction free and $\Gamma \vDash_S^{\mathfrak{S}_\infty} \varphi$, it follows that $\vDash_{S\cup\Gamma^+}^{\mathfrak{S}_\infty} \varphi$. We will show that $\vDash_S^{\mathfrak{S}_\infty} \varphi \Leftrightarrow S^* \vdash_{IPC} \varphi$. We will not prove the base case: $\vdash_S p \Leftrightarrow S^* \vdash_{IPC} p$. The proof is an induction of the level of atomic rules which is long but not difficult. The conjunction case is obvious. The implication case is as follows: $\vDash_S \varphi \to \psi \Leftrightarrow \varphi \vDash_S \psi \Leftrightarrow \vDash_{S\cup\{\varphi^+\}} \psi \Leftrightarrow S^* \cup \{\varphi^{+*}\} \vdash_{IPC} \psi \Leftrightarrow S^* \cup \{\varphi\} \vdash_{IPC} \psi \Leftrightarrow S^* \vdash_{IPC} \varphi \to \psi$. □

From which we get via Corollary 2.12:

Theorem 6.2. $L_{\mathfrak{S}_\infty} = L_{GInq} = IPC + GKP$.[7]

We have now demonstrated that the logic of proof-theoretic validity is general inquisitive logic. As we mentioned in the introduction this result is important both because it shows that general inquisitive logic has a proof-theoretic semantics but also because it provides an interesting answer to which logic is justified by proof-theoretic validity. What remains is to define a proof-theoretic system for inquisitive logic. It turns out we have a general mechanism for defining new proof-theoretic systems which satisfy sets of disjunction-free formulas:

Definition 6.3. Given a set of disjunction free formulas Δ let $\mathfrak{S}_\Delta = \{S \in \mathfrak{S}_\infty \mid \Delta^+ \subseteq S\}$

Lemma 6.4. For any set of disjunction free formulas Δ it follows that $\mathfrak{S}_\Delta$ is a filter on $\mathfrak{S}_\infty$, that $\vDash_{\mathfrak{S}_\Delta}$ has the disjunction property, and $\vDash_{\mathfrak{S}_\Delta} \Delta$.

Proof. Clearly $\mathfrak{S}_\Delta$ is a filter on $\mathfrak{S}_\infty$ so by Lemma 5.4 we have for all $S \in \mathfrak{S}_\Delta$ that $\vDash_S^{\mathfrak{S}_\infty} \Delta \Rightarrow \vDash_S^{\mathfrak{S}_\Delta} \Delta$. Note that for all $S \in \mathfrak{S}_\infty$, $\Delta \vDash_S^{\mathfrak{S}_\infty} \Delta$. So, by Lemma 5.8 it follows that $\vDash_{S\cup\Delta^+}^{\mathfrak{S}_\infty} \Delta$. Let $S \in \mathfrak{S}_\Delta$, it follows that $S' \in \mathfrak{S}_\infty$ and that $S' = S' \cup \Delta^+$. So $\vDash_{S'\cup\Delta^+}^{\mathfrak{S}_\infty} \Delta$ which implies $\vDash_{S'\cup\Delta^+}^{\mathfrak{S}_\Delta} \Delta$ and so $\vDash_{S'}^{\mathfrak{S}_\Delta} \Delta$. From which it follows that $\vDash_{\mathfrak{S}_\Delta} \Delta$. That $\vDash_{\mathfrak{S}_\Delta}$ has the disjunction property follows by observing that the proof of Lemma 5.11 can be easily transferred to a filter on $\mathfrak{S}_\infty$. □

If we consider the set of formulas true on a particular $\mathfrak{S}_\Delta$, sometimes this will be the set of all formulas because $\bot$ is among them. Of the consistent sets, many will not be closed under substitution. Some of these will include formulas which cannot be proven in classical logic, for example, $\mathfrak{S}_{\{p\}}$. This makes it clear that these systems can be weak logics that are either intermediate or stronger than classical logic.[8]

We have now collected all the necessary pieces to define quasi-PTV.

---

[7] I would like to thank an anonymous reviewer for pointing me towards Punčochář 2016 and suggesting this result.

[8] They cannot be classical as they all have the disjunction property which CPL does not.

Definition 6.5. Define $\mathfrak{S}_Q$ (Q for Quasi) based on $\mathfrak{S}_\infty$ as follows:

$$\mathfrak{S}_Q = \mathfrak{S}_{\{\neg\neg p \to p | p \text{ atomic}\}} = \{S \in \mathfrak{S}_\infty \mid ([p/\bot]\bot/p) \in S \text{ for all atomic } p\} \tag{18}$$

As $\mathfrak{S}_Q$ is a filter on $\mathfrak{S}_\infty$ we can show it satisfies the Kreisel-Putnam axiom as $\mathfrak{S}_\infty$ does. This is the last piece needed to show that $\mathfrak{S}_Q$ satisfies a condition from Corollary 2.11.

Lemma 6.6. $\mathfrak{S}_Q$ is a filter on $\mathfrak{S}_\infty$ and $\mathfrak{S}_Q$ satisfies the disjunction property, the Kreisel-Putnam axiom and $\neg\neg p \to p$ for all atomic p.

Proof. As $\mathfrak{S}_Q = \mathfrak{S}_{\{\neg\neg p \to p | p\text{atomic}\}}$ it follows by Lemma 6.4 that $\mathfrak{S}_Q$ is a filter on $\mathfrak{S}_\infty$, that $\mathfrak{S}_Q$ satisfied the disjunction property, and $\vDash_{\mathfrak{S}_Q} \neg\neg p \to p$ for all atomic p. As $\mathfrak{S}_Q$ is a filter on $\mathfrak{S}_\infty$ it follows by Lemma 5.4 that $\mathfrak{S}_Q$ proves everything $\mathfrak{S}_\infty$ does. By Lemma 5.10 and Lemma 5.12 it follows that $\mathfrak{S}_\infty$ satisfies the Kreisel-Putnam axiom and so, so does $\mathfrak{S}_Q$. □

This proof shows that $\mathfrak{S}_Q$ defines a logic $L_{\mathfrak{S}_Q} = \{\varphi \mid \vDash_{\mathfrak{S}_Q} \varphi\}$ which meets the requirements found in Corollary 2.11. By Lemmas 3.7 and 3.14 we have that $L_{\mathfrak{S}_Q}$ is the same logic as inquisitive logic. That is:

Theorem 6.7. $L_{\mathfrak{S}_Q} = L_{Inq}$

As we are working with weak logics we can ask the question what logic characterises the fragment closed under substitution. To do this we define a logics schematic fragment.

Definition 6.8. If L is a weak logic, its *schematic fragment* is:

$$Schm(L) = \{\varphi \in L \mid \text{ for all atomics } p_1, \dots, p_n \text{ in } \varphi \text{ and all } \varphi_1, \dots, \varphi_n, \text{ let } \varphi[p_1/\varphi_1, \dots, p_n/\varphi_n] \in L\}$$

That is $Schm(L)$ is every formula of L for which substitution is valid. If L is a weak intermediate logic then $Schm(L)$ is an intermediate logic. It follows from these results that the schematic fragment of both logics is Medvedev's logic.

Definition 6.9. Medvedev's logic of finite problems ML is defined semantically as the logic of all finite frames of the form $(\mathcal{P}(X) - \{X\}, \subseteq)$.

Ciardelli and Roelofsen (2011) have shown that the schematic fragment of L_{Inq} is Medvedev's logic of finite problems discussed above, which is not known to be axiomatizable. Punčochář (2016) has generalised this result based on work by Miglioli et al. (1989).[9] From this it follows that: $Schm(L_{\mathfrak{S}_Q}) = Schm(L_{\mathfrak{S}_\infty}) = ML$.

7 Conclusion

To summarise: We have seen that inquisitive semantics has the disjunction property, the generalised Kreisel-Putnam axiom and double negation elimination for atomic formulas, while general inquisitive semantics has the disjunction property, the generalised Kreisel-Putnam axiom and shares a disjunction free fragment with IPC. We have generated a system of PTV, namely quasi-PTV or $\mathfrak{S}_Q$, which is extensionally equivalent to inquisitive semantics and have shown that PTV or $\mathfrak{S}_\infty$ is extensionally equivalent to general inquisitive semantics.

[9] It is worth noting that Miglioli et al. 1989 gives the result without proof.

This answers an important question about proof-theoretic validity as presented in the works of Piecha, de Campos Sanz, and Schroeder-Heister. It also highlights a fascinating connection between inquisitive semantics and proof-theoretic semantics. Despite very different motivations, this connection suggests that inquisitive semantics can be given a constructive justification and that proof-theoretic validity might have wider applicability than previously though.

It remains open what logic is captured by $\mathfrak{S}_n$ for $n \in \mathbb{N}$. These logics are going to be less well behaviours than the ones considered in this paper because they will have the generalised Kreisel-Putnam axiom for formulas only below a certain complexity. They may also validate additional axioms as is seen by $\mathfrak{S}_1$'s having double negation elimination for atomic formulas (Piecha 2016).

References

Ciardelli, Ivano, Jeroen Groenendijk, and Floris Roelofsen (2018). *Inquisitive Semantics*. Oxford Surveys in Semantics and Pragmatics. Oxford: Oxford University Press.

Ciardelli, Ivano and Floris Roelofsen (Feb. 2011). "Inquisitive Logic". In: *Journal of Philosophical Logic* 40.1, pp. 55–94.

Dummett, Michael (1991). *The Logical Basis of Metaphysics*. Cambridge: Harvard University Press.

Gentzen, G. (1935). "Untersuchungen über das logische Schließen". In: *Mathematische Zeitschrift* 39, pp. 176–210, 405–431.

Groenendijk, Jeroen and M J B Stokhof (1984). "Studies on the Semantics of Questions and the Pragmatics of Answers". In:

Karttunen, Lauri and Stanley Peters (1980). "Interrogative quantifiers". In: *Time, tense, and quantifiers* 83, pp. 181–205.

Kleene, Stephen Cole (1952). *Introduction to metamathematics*. Vol. 483. New York: van Nostrand.

Kolmogorov, A. N. (1932). "Zur Deutung der Intuitionistischen Logik". In: *Mathematische Zeitschrift* 35, pp. 58–65. Trans. by Paolo Mancosu as "On the Interpretation of Intuitionistic Logic" (New York: Oxford University Press, 1998).

Maksimova, Larisa (1986). "On maximal intermediate logics with the disjunction property". In: *Studia Logica* 45, pp. 69–75.

Miglioli, P. et al. (1989). "Some results on intermediate constructive logics". In: *Notre Dame Journal of Formal Logic* 30(4), pp. 543–562.

Piecha, Thomas (2016). "Completeness in Proof-Theoretic Semantics". In: *Advances in Proof-Theoretic Semantics*. Ed. by Thomas Piecha and Peter Schroeder-Heister. Cham: Springer International Publishing, pp. 231–251.

Piecha, Thomas, Wagner de Campos Sanz, and Peter Schroeder-Heister (June 2015). "Failure of Completeness in Proof-Theoretic Semantics". In: *Journal of Philosophical Logic* 44.3, pp. 321–335.

Piecha, Thomas and Peter Schroeder-Heister (Feb. 2019). "Incompleteness of Intuitionistic Propositional Logic with Respect to Proof-Theoretic Semantics". In: *Studia Logica* 107.1, pp. 233–246.

Prawitz, Dag (1971). "Ideas and Results in Proof Theory". In: *Proceedings of the Second Scandinavian Logic Symposium*. Ed. by J.E. Fenstad. Vol. 63. Studies in Logic and the Foundations of Mathematics. Elsevier, pp. 235–307.

— (1973). "Towards A Foundation of A General Proof Theory". In: *Proceedings of the Fourth International Congress for Logic, Methodology and Philosophy of Science, Bucharest, 1971*. Ed. by

Patrick Suppes et al. Vol. 74. Studies in Logic and the Foundations of Mathematics Supplement C. Elsevier, pp. 225–250.

Punčochář, Vıt (Aug. 2016). "A Generalization of Inquisitive Semantics". In: *J. Philos. Logic* 45.4, pp. 399–428.

Schroeder-Heister, Peter (1984). "A Natural Extension of Natural Deduction". In: *J. Symbolic Logic* 49.4, pp. 1284–1300.

— (Feb. 2006). "Validity Concepts in Proof-theoretic Semantics". In: *Synthese* 148.3, pp. 525–571.

— (2018). "Proof-Theoretic Semantics". In: *The Stanford Encyclopedia of Philosophy.* Ed. by Edward N. Zalta. Spring 2018. Metaphysics Research Lab, Stanford University.

Stalnaker, Robert C (1976). "Possible Worlds". In: *Noûs* 10.1, pp. 65–75.

Troelstra, A. S. and Dirk van Dalen (1988). *Constructivism in Mathematics. An Introduction. Volume 2.* Amsterdam: Elsevier.